Integral Equations

MSc Course / 1st Semester / 2025-2026

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Chapter 1

Introductory Concepts

1.1 Definitions

An integral equation is an equation in which the unknown function $u(x)$ to be determined appears under the integral sign. A typical form of an integral equation in $u(x)$ is of the form

$$u(x) = f(x) + \int_{\alpha(x)}^{\beta(x)} K(x,t)\,u(t)dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where $K(x, t)$ is called the *kernel* of the integral equation, and $\alpha(x)$ and $\beta(x)$ are the limits of integration. In (1), it is easily observed that the unknown function $u(x)$ appears under the integral sign as stated above, and out of the integral sign in most other cases as will be discussed later. It is important to point out that the kernel $K(x, t)$ and the function $f(x)$ in (1) are given in advance. Our goal is to determine $u(x)$ that will satisfy (1), and this may be achieved by using different techniques that will be discussed in the forthcoming chapters. The primary concern of this text will be focused on introducing these methods and techniques supported by illustrative and practical examples.

Integral equations arise naturally in physics, chemistry, biology and engineering applications modelled by *initial value problems* for a finite interval $[a, b]$. More details about the sources and origins of integral equations can be found in [1] and [2]. In the following example we will discuss how an initial value problem will be converted to the form of an integral equation.

Example 1. Consider the initial value problem

$$u'(x) = 2xu(x) \qquad x \geq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

subject to the initial condition

$$u(0) = 1 \quad \ldots\ldots\ldots\

2. Volterra integral equations
3. Integro-differential equations
4. Singular integral equations
5. Volterra-Fredholm integral equations
6. Volterra-Fredholm integro-differential equations

In the following we will outline the basic definitions and properties of each type.

1.2.1 Fredholm Linear Integral Equations

The standard form of *Fredholm* linear integral equations, where the limits of integration *a* and *b* are constants, are given by the form

$$\phi(x)u(x) = f(x) + \lambda \int_a^b K(x,t)\,u(t)dt \ , \ \ a \le x,t \le b \ldots (7)$$

where the kernel of the integral equation, $K(x, t)$, and the function $f(x)$ are given in advance, and λ is a parameter. The equation (7) is called *linear* because the unknown function $u(x)$ under the integral sign occurs linearly, i.e. the power of $u(x)$ is one.

The value of $\phi(x)$ will give the following kinds of *Fredholm* linear integral equations:

1. When $\phi(x) = 0$, Eq. (7) becomes

$$f(x) + \lambda \int_a^b K(x,t)\,u(t)dt = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

and the integral equation is called Fredholm integral equation of the **first kind**.

2. When $\phi(x) = 1$, Eq. (7) becomes

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\,u(t)dt \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

and the integral equation is called *Fredholm* integral equation of the **second kind**.

In fact, the equation (9) can be obtained from (7) by dividing both sides of (7) by $\phi(x)$ provided that $\phi(x) \neq 0$.

In summary, the *Fredholm* integral equation is of the ***first kind if the unknown function u(x) appears only under the integral sign***. However, the *Fredholm* integral equation is of the ***second kind if the unknown function u(x) appears inside and outside the integral sign***.

1.2.2 Volterra Linear Integral Equations

The standard form of *Volterra* linear integral equations, where the limits of integration are ***functions of x rather than constants***, are of the form

$$\phi(x)u(x) = f(x) + \lambda \int_a^x K(x,t)\, u(t)dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

where the unknown function $u(x)$ under the integral sign occurs linearly as stated before.

It is worth noting that (10) can be viewed as ***a special case of Fredholm*** integral equation when the kernel $K(x, t)$ vanishes for $t > x$, x is in the range of integration $[a, b]$.

As in *Fredholm* equations, *Volterra* integral equations fall under two kinds, depending on the value of $\phi(x)$, namely:

1. When $\phi(x) = 0$, Eq. (10) becomes

$$f(x) + \lambda \int_a^x K(x,t)\, u(t)dt = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

and in this case the integral equation is called *Volterra* integral equation of the **first kind**.

2. When $\phi(x) = 1$, Eq. (10) becomes

$$u(x) = f(x) + \lambda \int_a^x K(x,t)\, u(t)dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

and in this case the integral equation is called *Volterra* integral equation of the **second kind**.

Examining the equations (7)–(12) carefully, the following remarks can be concluded:

In summary, the *Volterra* integral equation is of the *first kind* if the unknown function $u(x)$ appears only under the integral sign. However, the *Volterra* integral equation is of the *second kind* if the unknown function $u(x)$ appears inside and outside the integral sign.

Remarks

Examining the equations (7)–(12) carefully, the following remarks can be concluded.

1. **The structure of *Fredholm* and *Volterra* equations:**

The unknown function $u(x)$ appears *linearly* only under the integral sign in linear *Fredholm* and *Volterra* integral equations of the **first kind**. However, the unknown function $u(x)$ appears linearly **inside** the integral sign and **outside** the integral sign as well in the **second kind** of both linear *Fredholm* and *Volterra* integral equations.

2. **The limits of integration:**

In *Fredholm* integral equations, the **integral** is taken over a finite interval with **fixed** limits of integration. However, in *Volterra* integral equations, **at least** one limit of the range of integration is a **variable**, and the upper limit is the most commonly used with a variable limit.

3. **The origins of integral equations:**

It is important to note that integral equations arise in engineering, physics, chemistry, and biology problems [1] and [2]. Further, integral equations arise as *representation forms* of *differential equations*. Furthermore, *Fredholm* and *Volterra* integral equations arise from different origins and applications, such as *boundary value problems* as in *Fredholm* equations, and from *initial value problems* as in *Volterra* equations. Based on the fact that integral equations arise from distinct origins, different techniques and approaches will be used to determine the solution of each type of integral equations.

4. **The linearity property:**

As indicated before, the unknown function $u(x)$ in *Fredholm* and *Volterra* integral equations (9) and (12) occurs to the ***first power*** wherever it exists. However, nonlinear Fredholm and Volterra integral equations arise if $u(x)$ is replaced by a nonlinear function $F(u(x))$, such as $u^2(x)$, $\cos u(x)$, $\cosh u(x)$, $e^{u(x)}$ and so on. The following are examples of *nonlinear* integral equations:

$$u(x) = f(x) + \lambda \int_a^x K(x,t)\, u^2(t)dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

$$u(x) = f(x) + \lambda \int_a^x K(x,t)\, e^{u(t)}dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (14)$$

$$u(x) = f(x) + \lambda \int_0^1 K(x,t)\, \sin\big(u(t)\big)\, dt \quad \ldots\ldots\ldots\ldots\ldots\ldots . . (15)$$

where the linear function $u(x)$ in (1) has been replaced by the nonlinear functions $u^2(t), e^{u(t)}$, and $\sin(u(t))$ respectively.

5. **The homogeneity property:**

If we set $f(x) = 0$ in *Fredholm* or *Volterra* integral equation of the **second kind** given by (9) and (12), the resulting equation is called a ***homogeneous*** integral equation, otherwise it is called ***nonhomogeneous*** or ***inhomogeneous*** integral equation.

6. **The singular behavior of the integral equation:**

An integral equation is called ***singular*** if the integration is ***improper***. This usually occurs if the interval of integration is ***infinite***, or if the kernel becomes ***unbounded*** at one or more points of the interval of consideration $a \leq t \leq b$. Singular integral equations and the methods to handle them will be defined later. It is worth noting that **four** other types of integral equations, related to the two main classes *Fredholm* and *Volterra* integral equations arise in many science and engineering applications. In the following, we introduce these significant equations as distinct types.

References

[1] R. Estrada and R. Kanwal, Singular Integral Equations, Birkhauser, Berlin, (2000).

[2] J. H. He, Some Asymptotic Methods for Strongly Nonlinear Equations, International Journal of Modern Physics B, Vol. 20, No. 10 (2006) 1141–1199.

Chapter 1(Cont.)

Examples

1. Each of the following is example of **singular integral equation** (SIE):

$$a)\ u(x) = f(x) + \lambda \int_{-\infty}^{\infty} K(x,t)\, u(t)dt$$

$$b)\ u(x) = f(x) + \lambda \int_{0}^{x} \frac{1}{(x-t)^{\alpha}} \mathrm{u(t)}dt \quad , \qquad 0 < \alpha < 1$$

The following (*c*,*d*,*e*) are examples where the singular behavior in these examples has resulted from the range o integration becoming *infinite*.

$$c)\ u(x) = 2x + 6 \int_{0}^{\infty} sin(x-t)\, u(t)dt$$

$$d)\ u(x) = x + \frac{1}{3} \int_{-\infty}^{0} cos(x+t)\, u(t)dt$$

$$e)\ u(x) = 1 + x^2 + \frac{1}{6} \int_{-\infty}^{\infty} (x+t)\, u(t)dt$$

The following (*f*,*g*,*h*) are examples where the singular behavior in these examples has resulted from the kernel *K*(*x*, *t*) becomes *infinite* at one or more points in the domain of integration.

$$f)\ x = \int_{0}^{x} \frac{1}{(x-t)^{\alpha}} \mathrm{u(t)}dt \quad , \qquad 0 < \alpha < 1$$

$$g)\ x^2 = \int_{0}^{x} \frac{1}{\sqrt{x-t}} \mathrm{u(t)}dt$$

$$h)\ u(x) = 1 - 2\sqrt{x} - \int_0^x \frac{1}{\sqrt{x-t}}\mathrm{u(t)}dt$$

2. Each of the following is example of **<u>homogeneous linear integral equation</u>** (HLIE):

$$a)\ u(x) = \lambda \int_0^1 (1 - 3xt)\, u(t)dt$$

$$b)\ u(x) = \lambda \int_0^x (x - t)\, u(t)dt$$

$$c)\ u(x) = \lambda \int_0^1 K(x,t)\, u(t)dt\,, \qquad where$$

$$K(x,t) = \begin{cases} x(1-t) & 0 \le x \le t \\ t(1-x) & t \le x \le 1 \end{cases}$$

3. The IEs $f(x) = \int_a^b K(x,t)\, u(t)dt$ *and* $f(x) = u(x) + \lambda \int_a^b K(x,t)\, u(t)dt$

are the Fredholm IEs of the 1st and 2nd kind (respectively) if the kernel $K(x,t)$ satisfies one of the following conditions:

(i) $K(x,t)$ is continuous as a function of variables (x,t) in the square

$$\mathbb{III} = \{(x,t): a \le x \le b\ , \qquad a \le t \le b\}$$

(ii) $K(x,t)$ is maybe discontinuous for some (x,t) but the double integral

$$\int_a^b \int_a^b |K^2(x,t)|dxdt < \infty$$

$(takes\ a\ finite\ value: converges\ in\ \mathbb{III}\ by\ def.of\ Riemann)$

4. Consider a Fredholm IE of the second kind

$$x^2 = u(x) - \int_0^1 (x^2 + t^2)u(t)dt$$

where the kernel $K(x,t) = x^2 + t^2$ is ***defined*** and ***continuous*** in the square

$\mathbb{\Pi} = \{(x,t): 0 \le x \le 1\ ,\ 0 \le t \le 1\}$, $f(x) = x^2$ is a given function and $\lambda = -1$ is the parameter .

5. Consider a Fredholm IE of the second kind*;*

$$f(x) = u(x) + \int_0^1 ln|x-t|\, u(t)dt$$

where the kernel $K(x,t) = ln|x-t|$ is discontinuous (unbounded) in the square $\mathbb{\Pi} = \{(x,t): 0 \le x \le 1\ ,\ 0 \le t \le 1\}$, along the line $x = t$.
The function $f(x)$ is a given function and $\lambda = -1$ is the parameter.
Note that this IE is a Fredholm of the second kind because the double integral $\int_0^1 \int_0^1 ln^2|x-t|\, dxdt < \infty$, (i.e. takes a finite value in $\mathbb{\Pi}$).

Remark: Volterra equations may be considered a special case of Fredholm equations:
Suppose $a \le y \le b$ and put

$$K_1(x,y) = \begin{cases} K(x,y) & a \le y \le x \\ 0 & x < y \le b \end{cases}, \qquad then;$$

$$\int_a^b K_1(x,y)\, u(y)dy = \left(\int_a^x \quad + \int_x^b \right) K_1(x,y)u(y)dy = \int_a^x K(x,y)\, u(y)dy + 0$$

Integral Transforms

We have already known from (8) without loss of generality that the Fredholm IE of the first kind is

$$f(x) = \lambda \int_a^b K(x,t)\, u(t)dt\ ,\ \ where\ \phi(x) = 0\ and\ \lambda = -1.$$

This is an integral transform.

There are several different types of integral transforms. Particularly important examples of integral transforms include the *Fourier transform* and the *Laplace transform*, which we now discuss.

■ Fourier Transforms

A special case of a Fredholm equation of the first kind is

This is known as the Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy}\, u(y) dy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

Note that the kernel is complex in this case.

The solution to this equation is given by:

$$u(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy}\, f(x) dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

We will give the "physicist's" demonstration of this:

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x')\, dx' \int_{-\infty}^{\infty} e^{i(x-x')y}\, dy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

$$= \int_{-\infty}^{\infty} f(x')\, \delta(x - x') dx' = f(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

Here, we have used the fact that the Dirac "delta-function" may be written

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixy}\, dy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

(Please, I encourage you to demonstrate this, if you have not done so before).

Notice that the Fourier transform may be regarded as a limit of the Fourier series.
Let *f*(*x*) be expanded in a Fourier series in a box of size $\left[-\frac{L}{2}, \frac{L}{2}\right]$:

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \, e^{\frac{2\pi i n x}{L}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

We have chosen periodic boundary conditions here: $f\left(\frac{L}{2}\right) = f(-\frac{L}{2})$.

The a_n expansion coefficients may be determined for any given $f(x)$ using the orthogonality relations:

$$\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{\frac{2\pi i n x}{L}} e^{\frac{-2\pi i m x}{L}} dx = \delta_{mn} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

Hence,

$$a_n = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \, e^{\frac{-2\pi i m x}{L}} dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

Now consider taking the limit as $L \to \infty$. In this limit, the summation goes over to a continuous integral. Let $y = \frac{2\pi n}{L}$ and $g(y) = \frac{L a_n}{\sqrt{2\pi}}$. Then, using $dn = \left(\frac{L}{2\pi}\right) dy$,

$$f(x) = \lim_{L\to\infty} \sum_{n=-\infty}^{\infty} a_n \, e^{\frac{2\pi i n x}{L}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26)$$

$$= \lim_{L\to\infty} \sum_{n=-\infty}^{\infty} \frac{\sqrt{2\pi}}{L} g(y) e^{ixy} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} \, g(y) dy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28)$$

Furthermore:

$$g(y) = \frac{L a_n}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixy} \, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29)$$

We thus verify our earlier statements, including the δ-function equivalence, assuming our limit procedure is acceptable.

Suppose now that *f*(*y*) is an even function, $f(y) = f(-y)$.Then,

$$g(x) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{-ixy} f(y)dy + \int_{0}^{\infty} e^{-ixy} f(y)dy\right] \ldots\ldots\ldots(30)$$

$$g(x) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\left[e^{ixy} + e^{-ixy}\right] f(y)dy \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31)$$

$$g(x) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(y)\, cosxydy \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(32)$$

This is known as the **Fourier cosine transform**. It may be observed that the transform *g*(*x*) will also be an even function, and the solution for *f*(*y*) is:

$$f(y) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} g(x)\, cosxydx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(33)$$

Similarly, if *f*(*y*) is an odd function, we have the **Fourier sine transform**:

$$g(x) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(y)\, sinxydy \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(34)$$

where a factor of $-i$ has been absorbed. The solution for *f*(*y*) is

$$f(y) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} g(x)\, sinxydx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35)$$

Remark: We have seen (from one of the above examples) that if the kernel *K*(*x*, *y*) satisfies conditions such as square-integrability on [a; b] then convenient behavior is achieved for the solutions of the integral equation.

However, in the present case, we not only have $|a|, |b| \to \infty$, but the kernel e^{ixy} nowhere approaches zero. Thus, great care is required to ensure valid results.

■ Laplace Transforms

The Laplace transform is an integral transform of the form:

$$F(s) = \int_0^{\infty} f(x)\, e^{-sx} dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36)$$

with kernel $K(x,s) = K(s,x) = e^{-sx}$.

The "solution" for *f*(*x*) is:

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, e^{sx} ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (37)$$

where $x > 0$.

This transform can be useful for some functions where the Fourier transform does not exist. Problems at $x \to +\infty$ are removed by multiplying by e^{-cx}, where *c* is a positive real number. Then the problem at $-\infty$ is repaired by multiplying by the unit step function $\theta(x)$:

$$\theta(x) = \begin{cases} 1 & if\ x > 0 \\ 1/2 & if\ x = 0 \\ 0 & if\ x < 0 \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38)$$

Thus, we have

$$g(y) = \int_{-\infty}^{\infty} f(x)\, \theta(x) e^{-cx} e^{ixy} dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (39)$$

$$g(y) = \int_0^{\infty} f(x)\, e^{-cx} e^{ixy} dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (40)$$

where we have by convention also absorbed the $\frac{1}{\sqrt{2\pi}}$.

The inverse Fourier transform is just:

$$e^{-cx}\theta(x)f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(y)\, e^{ixy}dy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (41)$$

If we let $s = c + iy$ and define $F(s) = g(y)$ at $s = c + iy$, then

$$F(s) = \int_{0}^{\infty} f(x)\, e^{-sx}dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (42)$$

and

$$f(x)\theta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{x(c+iy)}dy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43)$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\, e^{sx}ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (44)$$

which is the above-asserted result.

<u>Some useful theorems for Fourier and Laplace transforms:</u>

Define

$$(\mathcal{F}f)(y) = g(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ixy}dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (45)$$

be the Fourier transform of *f*, and

$$(\mathcal{L}f)(s) = F(s) = \int_{0}^{\infty} f(x)\, e^{-sx}dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (46)$$

be the Laplace transform of *f*. Finally, let $\mathbb{T}$ ("transform") stand for either $\mathcal{F}$ or $\mathcal{L}$.

Then we can immediately verify the following properties: (check!!)

1. **Linearity**: For functions f and g and complex numbers α , β ,

$$\mathcal{T}(\alpha f + \beta g) = \alpha(\mathcal{T}f) + \beta(\mathcal{T}g) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (47)$$

2. **Transform of derivatives**: Integrate by parts to show that

$$(\mathcal{F}f')(y) = iy(\mathcal{F}f)(y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (48)$$

Assuming $f(x) \to 0 \; as \; x \to \pm\infty$, and

$$(\mathcal{L}f')(y) = s(\mathcal{L}f)(s) - f(0) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (49)$$

assuming $f(x) \to 0 \; as \; x \to \infty$ and defining $f(0) = \lim_{x\to 0+} f(x)$. The procedure here may be iterated to obtain expressions for **<u>higher derivatives</u>**. For example,

$$(\mathcal{L}f'')(y) = s^2(\mathcal{L}f)(s) - sf(0) - f'(0) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (49a)$$

3. **Transform of integrals:**

$$\left\{\mathcal{F}\left[\int f(x)dx\right]\right\}(y) = \frac{1}{iy}(\mathcal{F}f)(y) + C\delta(y) \ldots\ldots\ldots\ldots\ldots\ldots (50)$$

where *C* is an arbitrary constant arising from the arbitrary constant of integration of an indefinite integral;

$$\left\{\mathcal{L}\left[\int_0^x f(t)dt\right]\right\}(s) = \int_0^\infty e^{-sx}\,dx \int_0^x f(t)\,dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots (51)$$

$$= \int_0^\infty f(t)\,dt \int_t^\infty e^{-sx}\,dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (52)$$

$$= \frac{1}{s}(\mathcal{L}f)(s) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (53)$$

4. **Translation**:

$$[\mathcal{F}f(x+a)](y) = e^{iay}[\mathcal{F}f](y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (54)$$

$$[\mathcal{L}f(x+a)](s) = e^{as}[\mathcal{L}f](s) - \emptyset(a)\int_0^a f(x)e^{-sx}dx \ \ldots\ldots(55)$$

5. **Multiplication by an exponential**:

$$\{\mathcal{F}[e^{ax}f(x)]\}(y) = (\mathcal{F}f)(y+iy) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(56)$$

$$\{\mathcal{L}[e^{ax}f(x)]\}(s) = (\mathcal{L}f)(s-a) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(57)$$

6. **Multiplication by *x*:**

$$\{\mathcal{F}[xf(x)]\}(y) = i\frac{d}{dy}(\mathcal{F}f)(y) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(58)$$

$$\{\mathcal{L}[xf(x)]\}(s) = -\frac{d}{ds}(\mathcal{L}f)(s) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(59)$$

An important notion that arises in applications of integral transforms is that of a "*convolution*":

Definition: (Convolution): Given two functions $f_1(x)$ and $f_2(x)$, and constants *a*; *b*, the *convolution* of f_1 and $f_2$is:

$$g(x) = \int_a^b f_1(y)f_2(x-y)dy \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(60)$$

That is,

$$f_1(x) * f_2(x) = g(x) = \int_0^x f_1(y)\, f_2(x-y)dy \ \ldots\ldots\ldots\ldots\ldots\ldots(60a)$$

In the case of Fourier transforms, we are interested in – $a = b = \infty$. For Laplace transforms, $a = 0$, $b = \infty$. We then have the celebrated **convolution theorem**:

Theorem:

$$(\mathcal{F}g)(y) = \sqrt{2\pi}(\mathcal{F}f_1)(y)(\mathcal{F}f_2)(y) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(61)$$

$$(\mathcal{L}g)(y) = \sqrt{2\pi}(\mathcal{L}f_1)(y)(\mathcal{L}f_2)(y) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(62)$$

The proof is left as an **exercise**.

Examples:

1. Let $F(s) = (\mathcal{L}f)(x)$. Find the inverse Laplace transform of $F(s) = \frac{1}{s^2+9} + \frac{1}{s(s+1)}$ using a table of transform pairs:

$$f(x) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+9} + \frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}$$

$$= \frac{1}{3}sin3x + \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} = \frac{1}{3}sin3x + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \frac{1}{3}sin3x + 1 - e^{-x}$$

2. Find the Fourier transform of $g(x) = \frac{sinax}{x}$ using a table of integral transforms:

The *inverse* transform of $\frac{sina\lambda}{\lambda}$ is

$$f(x) = \begin{cases} 1/2 & |x| \le a \\ 0 & |x| > a \end{cases}$$

but since the Fourier transform is symmetrical, we know that

$$\mathcal{F}\{F(x)\} = 2\pi f(-\lambda)$$

Therefore, let

$$F(x) = \frac{sinax}{x} \quad and \quad f(\lambda) = \begin{cases} 1/2 & |x| \le a \\ 0 & |x| > a \end{cases}, \text{ so that}$$

$$\mathcal{F}\left\{\frac{sinax}{x}\right\} = 2\pi f(-\lambda) = \begin{cases} \pi & |\lambda| \le a \\ 0 & |\lambda| > a \end{cases}$$

■ **Mellin Transform**

The *Mellin transform* is an integral transform of a function *f*(*x*) defined on $(0,\infty)$.The general formula is:

$$M\{f\} = \int_0^\infty x^{\lambda-1} f(x)dx = F(\lambda) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (63)$$

If $x = e^{-t}$ and $f(x) = 0$ for $x < 0$ then the *Mellin transform* is reduced to a Laplace transform:

$$\int_0^\infty (e^{-t})^{\lambda-1} f(e^{-t})(-e^{-t})dt = -\int_0^\infty e^{-\lambda t} f(e^{-t})dt \dots\dots\dots\dots (64)$$

Like the inverse Laplace transform, the *inverse Mellin transform* involves complex integration, so tables of transform pairs are normally used to find both the Mellin transform of a function and its inverse.

When $F_1(\lambda) = M\{f_1\}$ and $F_2(\lambda) = M\{f_2\}$ then the convolution product is $M\{f_1 * f_2\} = F_1(\lambda)F_2(\lambda)$, where $f_1 * f_2 = \int_0^\infty \frac{f_1(t)f_2(\frac{x}{t})}{t} dt.$

Volterra integral equations with kernels of the form $K(\frac{x}{t})$ where the integration is performed on the interval $(0,\infty)$ may be solved using this method.

Example: Find the Mellin transform of $e^{-ax}, a > 0$. By definition:

$$M\{e^{-ax}\} = F(\lambda) = \int_0^\infty x^{\lambda-1} e^{-ax} dx$$

Let $ax = z$, so that

$$F(\lambda) = a^{-\lambda} \int_0^\infty z^{\lambda-1} e^{-z} dz$$

From the definition of the gamma function $\Gamma(v)$

$$\Gamma(v) = \int_0^\infty x^{v-1} e^{-x} dx$$

the result is

$$F(\lambda) = \frac{\Gamma(\lambda)}{a^{\lambda}}$$

■ Laplace's Method for Ordinary Differential Equations

As we have abovementioned, there are many integral transforms that we could investigate, but the Fourier and Laplace transforms are the most ubiquitous in physics applications. Rather than pursue other transforms, we'll look at another example that suggests the breadth of application of these ideas. *This is the "Laplace's Method" for the solution of ordinary differential equations*. This method represents a sort of generalization of the Laplace transform, using the feature of turning derivatives into powers.

Suppose we wish to solve the differential equation:

$$\sum_{k=0}^{n} (a_k + b_k x) f^{(k)}(x) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (65)$$

where we use the notation

$$f^{(k)}(x) = \frac{d^k f}{dx^k}(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (66)$$

In the spirit of the Laplace transform, assume a solution in the integral form

$$f(x) = \int F(s)\, e^{sx} ds, \quad on \;\; C \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (67)$$

where the contour "***C***" will be chosen to suit the problem (not necessarily the contour of the Laplace transform). We'll insert this proposed solution into the differential equation. Notice that

$$f'(x) = \int F(s)\, s e^{sx} ds, \quad on \;\; C \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (68)$$

Thus

$$0 = \int [U(s) + xV(x)]\, F(s) e^{sx} ds, \quad on\ C \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (69)$$

where

$$U(s) = \sum_{k=0}^{n} a_k\, s^k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (70)$$

$$V(s) = \sum_{k=0}^{n} b_k\, s^k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (71)$$

We may eliminate the *x* in the xV term of Eqn. 69 with an integration by parts:

$$\int [V(s)]\, F(s) x e^{sx} ds = [V(s)F(s)e^{sx}]_{c_1}^{c_2} - \int \frac{d}{ds}[V(s)F(s)] e^{sx} ds, \qquad (on\ C) \ldots (72)$$

where c_1 and c_2 are the endpoints of ***C***. Hence

$$0 = \int \left\{ U(s)F(s) - \frac{d}{ds}[V(s)F(s)] \right\} e^{sx} ds + [V(s)F(s)e^{sx}]_{c_1}^{c_2} \qquad (on\ C) \ldots (73)$$

We assume that we can choose C such that the integrated part vanishes. Then we will have a solution to the differential equation if

$$U(s)F(s) - \frac{d}{ds}[V(s)F(s)] = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (74)$$

Formally, we find a solution as:

$$\frac{d}{ds}[V(s)F(s)] = U(s)F(s) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (75)$$

$$V(s)\frac{dF(s)}{ds} = U(s)F(s) - F(s)\frac{dV(s)}{ds} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (76)$$

$$\frac{d\ lnF}{ds} = \frac{U}{V} - \frac{d \ln V}{ds} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (77)$$

$$lnF = \int \left(\frac{U}{V} - \frac{d \ln V}{ds} \right) ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (78)$$

$$= \int \frac{U}{V} ds - \ln V + \ln A \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (79)$$

where *A* is an arbitrary constant. Thus, the soluton for *F*(*s*) is;

$$F(s) = \frac{A}{V(s)} e^{\int^s \frac{U(s')}{V(s')} ds'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (80)$$

■ **Basic Numerical Integration Formulas**

General Equation:

$$u(x) = f(x) + \int_a^b K(x,t)\, u(t) dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (81)$$

Let

$$S_n(x) = \sum_{k=0}^{n} K(x, t_k)\, u(t_k) \Delta_k(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (82)$$

Let

$$u(x_i) = f(x_i) + \sum_{k=0}^{n} K(x_i, t_k)\, u(t_k) \Delta_k(t), \qquad i = 0,1, \ldots, n \quad \ldots (83)$$

If equal increments are used, then $\Delta x = \frac{b-a}{n}$.

♦ Midpoint Rule:

$$\int_a^b f(x) dx \approx \frac{b-a}{n} \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (84)$$

♦ Trapezoid Rule:

$$\int_a^b f(x) dx \approx \frac{b-a}{n} \left[\frac{1}{2} f(x_0) + \sum_{i=1}^{n-1} f(x_i) + \frac{1}{2} f(x_n)\right] \quad \ldots\ldots\ldots (85)$$

♦ Simpson's Rule:

$$\int_a^b f(x)dx \approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)] \ldots (86)$$

Chapter 2

1. Integro-Differential Equations

In the early 1900, Vito Volterra studied the phenomenon of population growth, and new types of equations have been developed and termed as the *integro-differential equations*. In this type of equations, the unknown function $u(x)$ occures in one side as an ***ordinary derivative*** and appears on the other side ***under the integral sign***.

Several phenomena in physics and biology [1] and [2] give rise to this type of integro-differential equations. Further, we point out that an integro-differential equation can be easily observed as an intermediate stage when we convert a differential equation to an integral equation as will be discussed later in the coming sections.

In the electrical engineering problem, the current $I(t)$ flowing in a closed circuit can be obtained in the form of the following integro-differential equation,

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I(\tau)\,d\tau = f(t), \;\; I(0) = I_0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.1)$$

where L is the inductance, R the resistance, C the capacitance, and $f(t)$ the applied voltage. Similar examples can be cited as follows:

$$u''(x) = f(x) + \lambda\int_0^x (x-t)\,u(t)dt \;\; , \;\; u(0) = 0 \;\; , u'(0) = 1 \ldots (2.2)$$

$$u'(x) = f(x) + \lambda\int_0^1 (xt)\,u(t)dt \;\; , \;\; u(0) = 1 \;\; \ldots\ldots\ldots\ldots\ldots\ldots . (2.3)$$

Equations (2.1) and (2.2) are of ***Volterra type integro-differential equations*** (VIDEq), whereas equation (2.3) ***Fredholm type integro-differential equations*** (FIDEq). These terminologies were concluded because of the presence of indefinite and definite integrals.

The following are examples of integro-differential equations:

$$u''(x) = -x + \int_0^x (x-t)\,u(t)dt \ , \ \ u(0) = 0 \ , u'(0) = 1 \ \ (\boldsymbol{VIDEq}) \ldots\ldots (2.4)$$

$$u'(x) = -sinx - 1 + \int_0^x u(t)\,dt \ , \ \ u(0) = 1 \ \ \ (\boldsymbol{VIDEq}) \ldots\ldots\ldots (2.5)$$

$$u'(x) = 1 - \frac{1}{3}x + \int_0^1 xt\,u(t)\,dt \ , \ \ u(0) = 1 \ \ \ (\boldsymbol{FIDEq}) \ldots\ldots\ldots . (2.6)$$

The solution for integro-differential equations will be established using in particular the most recent developed techniques.

2. Volterra-Fredholm Integral Equations

The *Volterra-Fredholm* integral equation, which is a combination of disjoint Volterra and Fredholm integrals, appears in one integral equation. The Volterra-Fredholm integral equations arise from the modelling of the spatiotemporal development of an epidemic, from boundary value problems and from many physical and chemical applications. The standard form of the Volterra-Fredholm integral equation reads

$$\boldsymbol{u(x) = f(x) + \int_0^x K_1(x,t)\,u(t)dt + \int_a^b K_2(x,t)\,u(t)dt} \ldots\ldots\ldots (2.7)$$

where $K_1(x, t)$ and $K_2(x, t)$ are the kernels of the equation.

Examples of the Volterra-Fredholm integral equations are

$$u(x) = 2x - \int_0^x (x-t)\,u(t)dt + \int_0^{\pi/2} x\,u(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots . (2.8)$$

and

$$u(x) = sinx - cosx - \int_0^x u(t)\,dt + \int_0^{\pi/2} u(t)\,dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.9)$$

Notice that the unknown function $u(x)$ appears inside the Volterra and Fredholm integrals and outside both integrals.

3. Volterra-Fredholm Integro-Differential Equations

The *Volterra-Fredholm integro-differential equation*, which is a combination of disjoint Volterra and Fredholm integrals and differential operator, may appear in one integral equation. The Volterra-Fredholm integro-differential equations arise from many physical and chemical applications similar to the Volterra-Fredholm integral equations. The standard form of the Volterra-Fredholm integro-differential equation reads

$$\boldsymbol{u^{(n)}(x) = f(x) + \int_0^x K_1(x,t)\, u(t)dt + \int_a^b K_2(x,t)\, u(t)dt} \ldots\ldots\ldots (2.10)$$

where $K_1(x, t)$ and $K_2(x, t)$ are the kernels of the equation, and "n" is the order of the ordinary derivative of $u(x)$. Notice that because this kind of equations contain ordinary derivatives, then initial conditions should be prescribed depending on the order of the derivative involved.

Examples of the Volterra-Fredholm integro-differential equations are

$$u'(x) = 1 + \int_0^x (x-t)\, u(t)dt + \int_0^1 xt\, u(t)dt \ , \qquad u(0) = 1 \ldots\ldots (2.11)$$

$$u''(x) = -x - \frac{1}{6}x^3 + \int_0^x u(t)\, dt + \int_{-\pi}^{\pi} x\, u(t)dt \ , \qquad u(0) = 0, \qquad u'(0) = 2 \ldots . (2.12)$$

Notice that the unknown function $u(x)$ appears inside the Volterra and Fredholm integrals and outside both integrals.

Exercises 1

a. Classify the following integral equation as Fredholm or Volterra integral equation, *linear* or *nonlinear* and *homogeneous* or *nonhomogeneous*:

$$u(x) = x - \frac{1}{6}x^3 + \int_0^x (x-t)\, u(t)dt$$

b. Classify the following integral equation as Fredholm or Volterra integral equation, *linear* or *nonlinear* and *homogeneous* or *nonhomogeneous*:

$$u(x) = \frac{1}{2} + x - \int_0^1 (x-t)\, u^2(t)dt$$

c. Classify the following equation as Fredholm or Volterra integro-differential equation, and *linear* or *nonlinear*., *linear* or *nonlinear*:

$$u'(x) = 1 - \frac{1}{3}x^3 + \int_0^x t\, u(t)dt \ , \ u(0) = 0$$

d. Discuss the type, linearity and homogeneity of the following equation:

$$u(x) = 1 + \int_0^x t\, u(t)dt + \int_0^\pi \sin\left(u(t)\right) u(t)dt$$

Exercises 2

In exercises 1-10, classify each of the following integral equations as Fredholm or Volterra integral equation, *linear* or *nonlinear*, and *homogeneous* or *nonhomogeneous*:

1. $u(x) = x + \int_0^1 xt\, u(t)dt$
2. $u(x) = 1 + x^2 + \int_0^x (x-t)\, u(t)dt$
3. $u(x) = e^x + \int_0^x t\, u^2(t)dt$
4. $u(x) = \int_0^1 (x-t)^2\, u(t)dt$
5. $u(x) = \frac{2}{3}x + \int_0^1 xt\, u(t)dt$
6. $u(x) = \frac{3}{4}x + \frac{1}{5} + \int_0^1 (x-t)^3\, u(t)dt$
7. $u(x) = 1 + \frac{x}{4}\int_0^1 \frac{1}{(x-t)}\frac{1}{u(t)}dt$
8. $u(x) = \frac{1}{2}cosx + \frac{1}{2}\int_0^{\frac{\pi}{2}} cosx\, u(t)dt$
9. $u(x) = 1 + \int_0^x (x-t)^2\, u^2(t)dt$
10. $u(x) = 1 - \int_0^x (x-t)\, u(t)dt$

In exercises 11–15, classify the following integro-differential equations as Fredholm integro-differential equation or Volterra integro-differential equation. Also determine whether the equation is *linear* or *nonlinear*:

11. $u'(x) = 1 + \int_0^x e^{-2t}\, u^3(t)dt\ ,\ u(0) = 1$
12. $u'(x) = 1 - \frac{1}{3}x + \int_0^1 xt\, u(t)dt\ , u(0) = 0$
13. $u''(x) = \frac{1}{2}x^2 - \int_0^x (x - t)\, u^3(t)dt\ ,\ u(0) = 1, u'(0) = 0$
14. $u'''(x) = sinx - x + \int_0^{\frac{\pi}{2}} xt\, u'(t)dt\ ,\ u(0) = 1\ ,\ u'(0) = 0\ , u''(0) = -1$
15. $u'''(x) = \frac{-1}{12}x^4 + \int_0^x (x - t)\, u(t)dt\ , u(0) = u'(0) = 0\ , u''(0) = 2$

In exercises 16–20, integrate both sides of each of the following differential equations once from 0 to *x*, and use the given initial condition to convert to a corresponding integral equation or integro-differential equation. (Follow Exercise 1.(a))

16. $u'(x) = 4u(x)\ ,\ u(0) = 1$
17. $u'(x) = 3x^2u(x)\ ,\ u(0) = 1$
18. $u'(x) = u^2(x)\ ,\ u(0) = 4$
19. $u''(x) = 4xu^2(x)\ ,\ u(0) = 2\ , u'(0) = 1$
20. $u''(x) = 2xu(x)\ ,\ u(0) = 0\ , u'(0) = 1$

In exercises 21–24, discuss the type, linearity and homogeneity of the following equations

21. $u(x) = x + \int_0^x \tan\,(u(t))\, u(t)dt - \int_0^1 (\mathrm{x} - \mathrm{t})\, u(t)dt$
22. $u'(x) = 1 + \int_0^x (\mathrm{x} - \mathrm{t})\, u(t)dt + \int_{-1}^1 \mathrm{xt}\, u(t)dt,\ u(0) = 1$
23. $u'(x) = sinx + \int_0^x t\, u(t)dt + \int_0^1 \mathrm{t}^2u^2\,(t)dt\ , u(0) = 0$
24. $u(x) = 1 + x + \int_0^x \frac{1}{\sqrt{x-t}}u^2(t)dt$

4. Solution of an Integral Equation

A solution of an *integral equation* or an *integro-differential equation* on the interval of integration is *a function u*(*x*) such that *it satisfies the given equation*. In other words, if the given solution is substituted in the right hand side of the equation, the output of this direct substitution must yield the left hand side, i.e. we should verify that the given function *u*(*x*) satisfies the integral equation or the integro-differential equation under discussion. This important concept will be illustrated first by examining the following examples.

Example 1. Show that $u(x) = e^x$ is a solution of the Volterra integral equation

$$u(x) = 1 + \int_0^x u(t)dt \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.13)$$

Solution: Substituting $u(x) = e^x$ in the right hand side (RHS) of (2.13) yields

$$RHS = 1 + \int_0^x e^t dt = 1 + [e^t]_0^x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.14)$$
$$= e^x = u(x) = LHS$$

Example 2. Show that $u(x) = x$ is a solution of the following Fredholm integral equation

$$u(x) = \frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\int_0^1 (x + t)u(t)dt \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.15)$$

Solution: Substituting $u(x) = x$ in the right hand side (RHS) of (2.15) yields

$$RHS = \frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\left[\frac{xt^2}{2} + \frac{t^3}{3}\right]_0^1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.16)$$
$$= x = u(x) = LHS$$

Example 3. Show that $u(x) = x$ is a solution of the following Fredholm integro-differential equation

$$u'(x) = \frac{2}{3} + \int_0^1 tu(t)dt \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.17)$$

Solution: Substituting $u(x) = x$ in the right hand side (RHS) of (2.17) we obtain

$$RHS = \frac{2}{3} + \int_0^1 t^2 dt = \frac{2}{3} + \left[\frac{t^3}{3}\right]_0^1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.18)$$
$$= 1 = u'(x) = LHS$$

Example 4. Check if $u(x) = x + e^x$ is a solution of the following Fredholm integral equation

$$u''(x) = e^x - \frac{4}{3}x + \int_0^1 xtu(t)dt \ , u(0) = 1 \ , u'(0) = 2 \ldots\ldots (2.19)$$

Solution: Substituting $u(x) = x + e^x$ in the right hand side (RHS) of (2.19) we obtain

$$RHS = e^x - \frac{4}{3}x + \int_0^1 xt(t + e^t)dt = e^x = u''(x) = LHS$$

Example 5. If $u(x) = e^{-x^2}$ is a solution of the following Volterra integral equation

$$u(x) = 1 - \beta \int_0^x tu(t)dt \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.20)$$

Find β.

Solution: Substituting $u(x) = e^{-x^2}$ into both sides of (2.20) we obtain

$$e^{-x^2} = 1 - \frac{\beta}{2} + \frac{\beta}{2}e^{-x^2} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.21)$$

Solving for β gives

$$\beta = 2 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.22)$$

Example 6. $u(x) = x^2 + x^3$ is a solution of the Fredholm integral equation

$$u(x) = x^3 - x^2 - 2x + \beta \int_{-1}^1 (xt^2 + x^2t)u(t)dt \ \ldots\ldots\ldots\ldots\ldots (2.23)$$

Find β.

Solution: Substituting $u(x) = x^2 + x^3$ into both sides of (2.23) we obtain

$$x^2 + x^3 = x^3 - x^2 - 2x + \beta\left(\frac{2}{5}x^2 + \frac{2}{5}x\right) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.24)$$

Solving for β gives

$$\beta = 5 \quad \ldots\ldots\ldots\

$$u(x) = \frac{5}{6}x + \frac{1}{2}\int_0^1 xu^2(t)dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.28)$$

It was found that equation (2.28) has two real solutions given by

$$u(x) = x \ , 5x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.29)$$

and this can be justified through direct substitution. The ***uniqueness*** concept is not applicable for this example and for **many other nonlinear problems**. The nonlinear problems will be examined briefly in forthcoming chapters. **The uniqueness criteria for nonlinear problems is justified for only specific problems under specific conditions**. **Generally speaking, nonlinear problems, differential and integral, give more than one solutions**.

It is useful to point out that our main concern in this text will be on the **linear integral equations and the linear integro-differential equations only**. The nonlinear Fredholm and the nonlinear Volterra integral equations will be examined as well. In addition, **we will focus our study on equations with closed form solutions as in Examples 1 and 2**. Other cases that may lead to a series solution will be investigated as well, supported by the development of the reliable techniques that will be discussed later. Moreover, nonlinear integral equations will be investigated in its simplest forms. The recent developed methods presented powerful techniques, and therefore these methods have been carried out with promising results in linear and nonlinear equations.

Exercises 3

In exercises 1–10, verify that the given function is a solution of the corresponding integral or integro-differential equation:

1. $u(x) = x + \int_0^{\frac{1}{4}} u(t)dt \ , \ u(x) = x + \frac{1}{24}$
2. $u(x) = \frac{2}{3}x + \int_0^1 xtu(t)dt \ , \ u(x) = x$
3. $u(x) = x + \int_0^1 xtu^2(t)dt \ , \ u(x) = 2x$
4. $u(x) = x + \int_0^x (x-t)u(t)dt \ , \ u(x) = sinx$
5. $u(x) = 2coshx - xsinshx - 1 + \int_0^x tu(t)dt \ , \ u(x) = coshx$
6. $u(x) = x + \frac{1}{5}x^5 - \int_0^x tu^3(t)dt \ , \ u(x) = x$

7. $u'(x) = 2x - x^4 + \int_0^x 4tu(t)dt\ ,\ u(0) = 0\ , u(x) = x^2$
8. $u''(x) = xcosx - 2sinx + \int_0^x tu(t)dt\ ,\ u(0) = 0\ , u'(0) = 1, u(x) = sinx$
9. $\int_0^x (x-t)^2 u(t)dt = x^3\ ,\ u(x) = 3$
10. $\int_0^x (x-t)^{\frac{1}{2}} u(t)dt = x^{\frac{3}{2}}\ ,\ u(x) = \frac{3}{2}$

In exercises 11–14, find the unknown if the solution of each equation is given:

11. $u(x) = 1 - \frac{1}{2}f(x) - \int_0^x (x-t)u(t)dt\ ,\ u(x) = 2cosx - 1$
12. $u(x) = f(x) + \int_0^x (x-t)u(t)dt\ ,\ u(x) = e^x$
13. $u(x) = 1 - \alpha \int_0^x 3t^2 u(t)dt\ ,\ u(x) = e^{-x^3}$
14. $u(x) = f(x) - 1 + \int_0^{\frac{\pi}{2}} tu(t)dt\ ,\ u(x) = sinx$

5. Converting Volterra Equation to an ODE

In this section we will present **the technique** that converts Volterra integral equations of the second kind to equivalent differential equations. This may be easily achieved by applying the important **Leibniz Rule for differentiating an integral**. It seems reasonable to review the basic outline of the rule.

5.1 Differentiating any Integral: Leibniz Rule

To differentiate the integral $\int_{\alpha(x)}^{\beta(x)} G(x,t)dt$ with respect to *x*, we usually apply the useful Leibniz rule given by:

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} G(x,t)dt = G\big(x,\beta(x)\big)\frac{d\beta}{dx} - G\big(x,\alpha(x)\big)\frac{d\alpha}{dx} + \int_{\alpha(x)}^{\beta(x)} \frac{\partial G(x,t)}{\partial x}dt \ldots\ldots\ldots\ldots (2.30)$$

where $G(x,t)$ and $\frac{\partial G(x,t)}{\partial x}$ are continuous functions in the domain *D* in the *xt*-plane that contains the rectangular region *R*, $a \le x \le b\ , t_0 \le t \le t_1$, and the limits of integration $\alpha(x)$ and $\beta(x)$ are defined functions having continuous derivatives for $a < x < b$. We note that Leibniz rule is usually presented in most calculus books, and our concern will be on using the rule rather than its theoretical proof. The following examples are illustrative and will be

mostly used in the coming approach that will be used to convert Volterra integral equations to differential equations.

Example 1. Find

$$\frac{d}{dx}\int_0^x (x-t)^2\, u(t)dt$$

Solution: In this example, $\alpha(x) = 0$, $\beta(x) = x$, hence $\alpha'(x) = 0$, $\beta'(x) = 1$ and

$\frac{\partial G(x,t)}{\partial x} = 2(x-t)u(t)$. Using Leibniz rule (2.30), we find

$$\frac{d}{dx}\int_0^x (x-t)^2\, u(t)dt = \int_0^x 2(x-t)\, u(t)dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.31)$$

Example 2. Find

$$\frac{d}{dx}\int_0^x \sin(x-t)\, u(t)dt$$

Solution: In this example, $\alpha(x) = 0$, $\beta(x) = x$, hence $\alpha'(x) = 0$, $\beta'(x) = 1$ and

$\frac{\partial G(x,t)}{\partial x} = cos(x-t)u(t)$. Using Leibniz rule (2.30), we find

$$\frac{d}{dx}\int_0^x \sin(x-t)\, u(t)dt = \int_0^x \cos(x-t)\, u(t)dt + \left(\frac{dx}{dx}\right)\left(\sin(x-x)\, u(x)\right)$$

$$-\left(\frac{d0}{dx}\right)\left(\sin(x-0)\, u(0)\right)$$

$$= \int_0^x \cos(x-t)\, u(t)dt$$

Example 3. Find

$$\frac{d}{dx}\int_0^x (x-t)\,u(t)dt$$

Solution: In this example, $\alpha(x) = 0$, $\beta(x) = x$, hence $\alpha'(x) = 0$, $\beta'(x) = 1$ and

$\frac{\partial G(x,t)}{\partial x} = u(t)$. Using Leibniz rule (2.30), we find

$$\frac{d}{dx}\int_0^x (x-t)\,u(t)dt = \int_0^x u(t)\,dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.32)$$

Example 4. Find

$$\frac{d}{dx}\int_0^x u(t)\,dt$$

Solution: Proceeding as before, we find that

$$\frac{d}{dx}\int_0^x u(t)\,dt = u(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.33)$$

We now turn to ***our main goal*** to convert a Volterra integral equation to an equivalent differential equation. This can be easily achieved by differentiating both sides of the integral equation, noting that Leibniz rule should be used in differentiating the integral as stated above. The differentiating process should be continued as many times as needed until we obtain a pure differential equation with the integral sign removed. Moreover, the initial conditions needed can be obtained by substituting $x = 0$ in the integral equation and the resulting integro-differential equations as will be shown. We are now ready to give the following illustrative examples.

Example 4. Find the initial value problem equivalent to the Volterra integral equation

$$u(x) = 1 + \int_0^x u(t)dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.34)$$

Solution: Differentiating both sides of the integral equation and using Leibniz rule we find

$$u'(x) = \frac{d}{dx}\int_0^x u(t)dt = u(x) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.35)$$

The initial condition can be obtained by substituting $x = 0$ into both sides of the integral equation; hence we find $u(0) = 1$. Consequently, the corresponding initial value problem of first order is given by

$$u'(x) - u(x) = 0, \qquad u(0) = 1 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.36)$$

Example 6. Convert the following Volterra integral equation to an initial value problem

$$u(x) = x + \int_0^x (t - x)u(t)dt \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.37)$$

Solution: Differentiating both sides of the integral equation we obtain

$$u'(x) = 1 + \frac{d}{dx}\int_0^x (t - x)u(t)dt = 1 - \int_0^x u(t)dt \ \ldots\ldots\ldots\ldots (2.38)$$

We differentiate both sides of the resulting integro-differential equation (2.38) to remove the integral sign, therefore we obtain

$$u''(x) = -u(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.39)$$

or equivalently

$$u''(x) + u(x) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.40)$$

The related initial conditions are obtained by substituting $x = 0$ in $u(x)$ and in $u'(x)$ in the equations above, and as a result we find $u(0) = 0$ and $u'(0) = 1$. Combining the above results yields the equivalent initial value problem of the second order given by

$$u''(x) + u(x) = 0 \ , u(0) = 0 \ , u'(0) = 1 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.41)$$

with constant coefficients that can be easily handled.

Example 7. Find the initial value problem equivalent to the Volterra integral equation

$$u(x) = x^3 + \int_0^x (x-t)^2 u(t)dt \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.42)$$

Solution: Differentiating both sides of (2.42) three times we find

$$\begin{cases} u'(x) &= 3x^2 + 2\int_0^x (x-t)u(t)dt \\ u''(x) &= 6x + 2\int_0^x u(t)dt \\ u'''(x) &= 6 + 2u(x) \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.43)$$

The proper initial conditions can be easily obtained by substituting $x = 0$ in $u(x)$, $u'(x)$ and $u''(x)$ in the obtained equations above. Consequently, we obtain the non-homogeneous initial value problem of third order given by

$$u'''(x) - 2u(x) = 6 \ , \qquad u(0) = u'(0) = u''(0) = 0 \quad \ldots\ldots\ldots\ldots\ldots (2.44)$$

with constant coefficients that can be easily solved as an ordinary differential equation. **We point out here that the solution of initial value problems, that result from converting Volterra integral equations, will be discussed later.**

Exercises 4

In exercise 1-4, find $\frac{d}{dx}$ for the given integrals by using Leibniz rule:

1. $\int_0^x (x-t)^3 u(t)dt$
2. $\int_x^{x^2} e^{xt}dt$

3. $\int_0^x (x-t)^4 u(t)dt$
4. $\int_x^{4x} \sin(x+t)dt$

In exercise 5-14, convert each of the Volterra integral equations to an equivalent initial value problem:

5. $u(x) = 1 + x + \int_0^x (x-t)^2 u(t)dt$
6. $u(x) = e^x - \int_0^x (x-t)u(t)dt$
7. $u(x) = x + \int_0^x (x-t)u(t)dt$
8. $u(x) = x - cosx + \int_0^x (5 - 6x + 6t)u(t)dt$
9. $u(x) = 2 + 3x + 5x^2 + \int_0^x [1 + 2(x-t)]u(t)dt$
10. $u(x) = -5 + 6x + \int_0^x (x-t)u(t)dt$
11. $u(x) = tanx - \int_0^x u(t)dt \ \ , x < \frac{\pi}{2}$
12. $u(x) = 1 + x + \frac{5}{2}x^2 + \int_0^x \left[3 + 6(x-t) - \frac{5}{2}(x-t)^2\right] u(t)dt$
13. $u(x) = x^4 + x^2 + 2\int_0^x (x-t)^2 u(t)dt$
14. $u(x) = x^2 + \frac{1}{6}\int_0^x (x-t)^3 u(t)dt$

6. Converting IVP to Volterra Equations

In this section, we will study the method that converts an initial value problem to an equivalent Volterra integral equation. Before outlining the method needed, we wish to recall the useful transformation formula

$$\int_0^x \int_0^{x_1} \int_0^{x_2} \dots \int_0^{x_{n-1}} \mathbf{f(x_n)\, dx_n \dots dx_1} = \frac{\mathbf{1}}{\mathbf{(n-1)!}} \int_0^x \mathbf{(x-t)^{n-1} f(t) dt} \dots\dots\dots (\mathbf{2.45})$$

that converts any multiple integral to a single integral. This is an essential and useful formula that will be employed in the method that will be used in the conversion technique. We point out that this formula appears in most calculus texts. For practical considerations, the formulas

$$\int_0^x \int_0^x f(t)dtdt = \int_0^x (x-t)\, f(t)dt \ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2.46)$$

and

$$\int_0^x \int_0^x \int_0^x f(t)\, dtdtdt = \frac{1}{2!}\int_0^x (x-t)^2 f(t)dt \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2.47)$$

are two special cases of the formula given above, and the mostly used formulas that will transform double and triple integrals respectively to a single integral for each. For simplicity reasons, we prove the first formula (2.46) that converts double integral to a single integral. Noting that the right hand side of (2.46) is a function of *x* allows us to set the equation

$$I(x) = \int_0^x (x-t)\, f(t)dt \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . (2.48)$$

Differentiating both sides of (2.48), and using Leibniz rule we obtain

$$I'(x) = \int_0^x f(t)\, dt \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . (2.49)$$

Integrating both sides of (2.49) from 0 to *x*, noting that *I*(0) = 0 from (2.48) we find

$$I(x) = \int_0^x \int_0^x f(t)\, dtdt \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . (2.50)$$

Equating the right hand sides of (2.48) and (2.50) **completes the proof for this special case**. The proof for the conversion of the triple integral to a single integral given by (2.47) may be carried out in the same manner. The procedure of reducing multiple integral to a single integral will be illustrated by examining the following examples.

Example 1. Convert the following quadruple integral

$$I(x) = \int_0^x \int_0^x \int_0^x \int_0^x f(t)\, dtdtdtdt \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2.51)$$

to a single integral**.**

Solution: Using the formula (2.45), noting that *n* = 4, we find

$$I(x) = \frac{1}{3!}\int_0^x (x-t)^3\, u(t)dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.52)$$

the equivalent single integral.

Example 2. Convert the triple integral

$$I(x) = \int_0^x \int_0^x \int_0^x u(t)\, dtdtdt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.53)$$

to a single integral**.**

Solution: Using the formula (2.45), noting that *n* = 4, we find

$$I(x) = \frac{1}{2!}\int_0^x (x-t)^2\, u(t)dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.54)$$

the equivalent single integral.

Returning to the main goal of this section, we discuss the technique that will be used to convert an initial value problem to an equivalent Volterra integral equation. Without loss of generality, and for simplicity reasons, we apply this technique to a third order initial value problem given by

$$y'''(x) + p(x)y''(x) + q(x)y'(x) + r(x)y(x) = g(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.55)$$

subject to the initial conditions

$$y(0) = \alpha\, , y'(0) = \beta\, , y''(0) = \gamma \;;\; \alpha, \beta, and\ \gamma\ are\ constants \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.56)$$

The coefficient functions *p*(*x*) , *q*(*x*) and *r*(*x*) are analytic functions by assuming that these functions have Taylor expansions about the origin. Besides, we assume that *g*(*x*) is

continuous through the interval of discussion. To transform (2.55) into an equivalent Volterra integral equation, we first set

$$y'''(x) = u(x) \dots\dots\dots\

$$y'(x) = \beta + \gamma x + \int_0^x (x-t)u(t)\,dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.62)$$

and

$$y(x) = \alpha + \beta x + \frac{1}{2}\gamma x^2 + \frac{1}{2}\int_0^x (x-t)^2\, u(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.63)$$

respectively. Substituting (2.57), (2.59), (2.62) and (2.63) into (2.55) leads to the following Volterra integral equation of the second kind

$$u(x) = f(x) + \int_0^x K(x,t)u(t)dt \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.64)$$

Where

$$K(x,t) = p(x) + q(x)(x-t) + \frac{1}{2!}r(x)(x-t)^2 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.65)$$

and

$$f(x) = g(x) - \left\{\gamma p(x) + \beta q(x) + \alpha r(x) + \gamma x q(x) + r(x)\left(\beta x + \frac{1}{2}\gamma x^2\right)\right\} \ldots\ldots (2.66)$$

<u>The following examples will be used to illustrate the above discussed technique.</u>

<u>Example 3</u>. Convert the following initial value problem

$$y''' - 3y'' - 6y' + 5y = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.67)$$

subject to the initial conditions

$$y(0) = y'(0) = y''(0) = 1 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.68)$$

to an equivalent Volterra integral equation.

Solution: As indicated before, we first set

$$y'''(x) = u(x) \text{ … … …

the equivalent Volterra integral equation.

Example 4. Find the equivalent Volterra integral equation to the following initial value problem

$$y''(x) + y(x) = cosx, \qquad y(0) = 0 \ , y'(0) = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.76)$$

Solution: Proceeding as before, we set

$$y''(x) = u(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.77)$$

Integrating both sides of (2.77) from 0 to x, using the initial condition $y'(0) = 1$ yields

$$y'(x) = 1 + \int_0^x u(t)\, dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.78)$$

Integrating (2.78), using the initial condition $y(0) = 0$ leads to

$$y(x) = x + \int_0^x \int_0^x u(t)\, dtdt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.79)$$

or equivalently

$$y(x) = x + \int_0^x (x - t)\, u(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.80)$$

upon using the conversion rule (2.46). Inserting (2.77) and (2.80) into (2.76) leads to the following required Volterra integral equation

$$u(x) = cosx - x - \int_0^x (x - t)\, u(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.81)$$

the equivalent Volterra integral equation.

As previously remarked, linear Volterra integral equations will be discussed extensively later. It is of interest to point out that the newly developed methods and the traditional methods will be introduced too.

Exercises 5

In exercises 1-3, convert each of the following first order initial value problem to a Volterra integral equation:

1. $y' + y = 0\ ,\ y(0) = 1$
2. $y' - y = x\ ,\ y(0) = 0$
3. $y' + y = sec^2 x\ ,\ y(0) = 0$

In exercises 4-10, derive an equivalent Volterra integral equation to each of the following initial value problems of second order:

4. $y'' + y = 0, y(0) = 1, y'(0) = 0$
5. $y'' - y = 0, y(0) = 1, y'(0) = 1$
6. $y'' + 5y' + 6y = 0, y(0) = 1, y'(0) = 1$
7. $y'' + y' = 0, y(1) = 0, y'(1) = 1$
8. $y'' + y' - 2y = 2x, y(0) = 0, y'(0) = 1$
9. $y'' + y = \sin x, y(0) = 0, y'(0) = 0$
10. $y'' - \sin x\, y' + e^x y = x, y(0) = 1, y'(0) = -1$

In exercises 11-15, convert each of the following initial value problems of higher order to an equivalent Volterra integral equation:

11. $y''' - y'' - y' + y = 0, y(0) = 2, y'(0) = 0, y''(0) = 2$
12. $y''' + 4y' = x, y(0) = 0, y'(0) = 0, y''(0) = 1$

13. $y^{iv} + 2y'' + y = 3x + 4,\ y(0) = 0,\ y'(0) = 0,\ y''(0) = 1,\ y'''(0) = 1$

14. $y^{iv} - y = 0,\ y(0) = 1,\ y'(0) = 0,\ y''(0) = -1,\ y'''(0) = 0$

15. $y^{iv} + y'' = 2e^{x},\ y(0) = 2,\ y'(0) = 2,\ y''(0) = 1,\ y'''(0) = 1$

7. Converting BVP to Fredholm Equations

In the last section we have demonstrated how an IVP can be transformed to an equivalent Volterra integral equation. We present in this section how a boundary value problem (BVP) can be converted to an equivalent Fredholm integral equation.

The method is similar to that discussed in the previous section with some exceptions that are related to the boundary conditions. It is to be noted here that the method of reducing a BVP to a Fredholm integral equation is complicated and rarely used.

Example 1. Let us consider the following ***second-order ordinary differential*** with the given boundary conditions.

$$y''(x) + P(x)y'(x) + Q(x)y(x) = f(x) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (E_1)$$

with the boundary conditions

$$\begin{cases} x = a: \quad y(a) = \alpha \\ y = b: \quad y(b) = \beta \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, (E_2)$$

where α and β are given constants. Let us make transformation

$$y''(x) = u(x) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\

$$y(x) = y(a) + (x-a)y'(a) + \int_a^x \int_a^x u(t)\, dtdt$$

$$= \alpha + (x-a)y'(a) + \int_a^x \int_a^x u(t)\, dtdt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (E_5)$$

and using the boundary condition at $x = b$ yields

$$y(b) = \beta = \alpha + (b-a)y'(a) + \int_a^b \int_a^b u(t)\, dtdt$$

and the unknown constant $y'(a)$ is determined as

$$y'(a) = \frac{\beta - \alpha}{b-a} - \frac{1}{b-a} \int_a^b \int_a^b u(t)\, dtdt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (E_6)$$

Hence the solution (E_5) can be rewritten as

$$y(x) = \alpha + (x-a)\left\{\frac{\beta - \alpha}{b-a} - \frac{1}{b-a} \int_a^b \int_a^b u(t)\, dtdt\right\} + \int_a^x \int_a^x u(t)\, dtdt \ldots\ldots (E_7)$$

Therefore, equation (E_1) can be written in terms of *u*(*x*) as

$$u(x) = f(x) - P(x)\left\{y'(a) + \int_a^x u(t)dt\right\}$$

$$-Q(x)\left\{\alpha + (x-a)y'(a) + \int_a^x \int_a^x u(t)\, dtdt\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (E_8)$$

where $u(x) = y''(x)$ and so *y*(*x*) can be determined, in principle, from equation (E_7). This is a complicated procedure to determine the solution of a BVP by equivalent Fredholm integral equation.

<u>Remark</u>: A special case

If $a = 0$ and $b = 1$, i.e. $0 \le x \le 1$, then

$$y(x) = \alpha + xy'(0) + \int_0^x \int_0^x u(t)\, dtdt$$

$$= \alpha + xy'(a) + \int_0^x (x - t)u(t)\, dt$$

And hence the unknown constant $y'(0)$ can be determined as

$$y'(0) = (\beta - \alpha) - \int_0^1 (1 - t)u(t)\, dt$$

$$= (\beta - \alpha) - \int_0^x (1 - t)u(t)\, dt - \int_x^1 (1 - t)u(t)\, dt$$

And thus we have

$$u(x) = f(x) - P(x)\left\{y'(0) + \int_0^x u(t)dt\right\} - Q(x)\left\{\alpha + (x)y'(0) + \int_0^x (x - t)\, dt\right\}$$

$$u(x) = f(x) - (\beta - a)\{P(x) + xQ(x)\} - \alpha Q(x) + \int_0^1 K(x, t)u(t)\, dt \quad \ldots\ldots\ldots\ldots\ldots (E_9)$$

where the kernel $K(x, t)$ is given by

$$K(x, t) = \begin{cases} \{P(x) + tQ(x)\}(1 - x) & 0 \le t \le x \\ \{P(x) + xQ(x)\}(1 - t) & x \le t \le 1 \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (E_{10})$$

<u>Remark</u>: It can be easily verified that $K(x, t) = K(t, x)$ confirming that ***<u>the kernel is symmetric</u>***. ***<u>The Fredholm integral equation is given by</u> (E_9)***.

<u>A special attention should be taken to define $y'(0)$, since it is not always given</u>, as we have seen above. This can be easily determined from the resulting equations. It seems useful and practical to illustrate this method by applying it to an example. We demonstrate this method with an illustration.

Example 1. We ***want to derive*** an equivalent Fredholm integral equation to the following boundary value problem

$$y''(x) + y(x) = x\,, \qquad 0 < x < \pi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.82)$$

subject to the boundary conditions

$$y(0) = 1, \qquad y(\pi) = \pi - 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.83)$$

Solution: We first set

$$y''(x) = u(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.84)$$

Integrating both sides of (2.84) from 0 to x gives

$$\int_0^x y''(t)\,dt = \int_0^x u(t)\,dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.85)$$

or equivalently

$$y'(x) = y'(0) + \int_0^x u(t)\,dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.86)$$

As indicated earlier, $y'(0)$ is not given in this boundary value problem. However, $y'(0)$ will be determined later by using the boundary condition at $x = \pi$.

Integrating both sides of (2.86) from 0 to x and using the given boundary condition at $x = 0$ we find

$$y(x) = 1 + xy'(0) + \int_0^x (x - t)u(t)\,dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.87)$$

upon converting the resulting double integral to a single integral as discussed before.

It remains to evaluate $y'(0)$, and this can be obtained by substituting $x = \pi$ in both sides of (2.87) and using the boundary condition at $x = \pi$, hence we find

$$y(\pi) = 1 + \pi y'(0) + \int_0^\pi (\pi - t)u(t)\,dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.88)$$

Solving (2.88) for $y'(0)$ we obtain

$$y'(0) = \frac{1}{\pi}\left((\pi - 2) - \int_0^{\pi} (\pi - t)u(t)\, dt \right) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2.89)$$

Substituting (2.89) for $y'(0)$ into (2.87) yields

$$y(x) = 1 + \frac{x}{\pi}\left((\pi - 2) - \int_0^{\pi} (\pi - t)u(t)\, dt \right) + \int_0^{x} (x - t)u(t)\, dt \dots\dots\dots\dots (2.90)$$

Substituting (2.84) and (2.90) into (2.82) we get

$$u(x) = x - 1 - \frac{x}{\pi}\left((\pi - 2) - \int_0^{\pi} (\pi - t)u(t)\, dt \right) + \int_0^{x} (x - t)u(t)\, dt \dots\dots\dots\dots (2.91)$$

The following identity

$$\int_0^{\pi} . = \int_0^{x} . + \int_x^{\pi} . \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2.92)$$

will carry the equation (2.91) to

$$u(x) = x - 1 - \frac{x}{\pi}(\pi - 2) + \frac{x}{\pi}\int_0^{x} (\pi - t)u(t)\, dt + \frac{x}{\pi}\int_x^{\pi} (x - t)u(t)\, dt$$
$$- \int_0^{x} (x - t)u(t)\, dt \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2.93)$$

or equivalently, after performing simple calculations and adding integrals with similar limits

$$u(x) = \frac{2x - \pi}{\pi} - \int_0^{x} \frac{t(x - \pi)}{\pi} u(t)\, dt - \int_x^{\pi} \frac{x(t - \pi)}{\pi} u(t)\, dt \dots\dots\dots\dots\dots (2.94)$$

Consequently, the desired Fredholm integral equation of the second kind is given by

$$u(x) = \frac{2x - \pi}{\pi} - \int_0^{\pi} K(x,t)u(t)\,dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.95)$$

where the kernel $K(x, t)$ is defined by

$$K(x,t) = \begin{cases} \dfrac{t(x-\pi)}{\pi} & 0 \le t \le x \\ \dfrac{x(t-\pi)}{\pi} & x \le t \le \pi \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.96)$$

It is worth noting that the equation (2.95) obtained is a **nonhomogeneous Fredholm integral equation**, and this **(usually)** results when converting **a nonhomogeneous boundary value problem to its equivalent integral equation**. However, **homogeneous boundary value problems (always) lead to homogeneous Fredholm integral equations**. Further, we point out here that the solution of boundary value problems is much easier if compared with the solution of its corresponding Fredholm integral equation. This leads to the conclusion that transforming boundary value problem to Fredholm integral equation is less important if compared with transforming initial value problems to Volterra integral equations.

It is of interest to note that the recent developed methods, namely the **Adomian decomposition method** and **the direct computation method**, will be introduced later to handle Fredholm integral Equations. In addition, **the traditional methods**, namely the ***successive approximations method*** and ***the method of successive substitutions***, will be used in the following chapters as well.

Example 2. Let us consider the following boundary value problem.

$$\begin{cases} y''(x) = f(x, y(x)), & 0 \le x \le 1 \\ y(0) = y_0 \ , \quad y(1) = y_1 \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.97)$$

Integrating equation (2.97) with respect to x from 0 to x two times yields

$$y(x) = y(0) + xy'(0) + \int_0^x \int_0^x f(t, y(t))\,dtdt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.98)$$

$$= y_0 + xy'(0) + \int_0^x (x-t) f\big(t, y(t)\big)\, dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.99)$$

To determine the unknown constant $y'(0)$, we use the condition at $x = 1$, i.e. $y(1) = y_1$. Hence equation (2.99) becomes

$$y(1) = y_1 = y_0 + y'(0) + \int_0^1 (1-t) f\big(t, y(t)\big)\, dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.100)$$

and the value of $y'(0)$ is obtained as

$$y'(0) = (y_1 - y_0) - \int_0^1 (1-t) f\big(t, y(t)\big)\, dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.101)$$

Thus, equation (2.99) can be written as

$$y(x) = y_0 + x(y_1 - y_0) - \int_0^1 K(x,t) f\big(t, y(t)\big)\, dt\,, \qquad 0 \le x \le 1 \ldots\ldots\ldots (2.102)$$

Where

$$K(x,t) = \begin{cases} t(1-t) & 0 \le t \le x \\ x(1-t) & x \le t \le 1 \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.103)$$

Once again we can **reverse** the process and deduce that the function *y* which satisfies the integral equation also satisfies the BVP.

Remark: If we now specialize equation (2.97) of **Example2** to the simple linear BVP

$$y''(x) = -\lambda y(x), \qquad 0 < x < 1$$

with the boundary conditions,

$$y(0) = y_0, \qquad y(1) = y_1$$

then equation (2.102) reduces to the second kind Fredholm integral equation

$$y(x) = F(x) + \lambda \int_0^1 K(x,t)\, y(t)dt\,, \qquad 0 \le x \le 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.104)$$

Where

$F(x) = y_0 + x(y_1 - y_0)$.

Exercises 6

1. Reduce the following BVP

$$y''(x) + y(x) = x\ , \qquad 0 < x < \frac{\pi}{2}, \qquad y(0) = 1, y\left(\frac{\pi}{2}\right) = \pi$$

into Fredholm integral equation.

2. Derive the equivalent Fredholm integral equation for each of the following boundary value problems:

a) $y'' + 4y = \sin x,\ 0 < x < 1,\ y(0) = y(1) = 0$

b) $y'' + 2xy = 1,\ 0 < x < 1,\ y(0) = y(1) = 0$

c) $y'' + y = x,\ 0 < x < 1,\ y(0) = 1,\ y(1) = 0$

d) $y'' + y = x,\ 0 < x < 1,\ y(0) = 1,\ y'(1) = 0$

8. Types of solution techniques

There are a host of solution techniques that are available to solve the integral equations. Two important traditional methods are *the method of successive approximations* and *the method of successive substitutions*. In addition, the series method and the direct computational method are also suitable for some problems. The recently developed methods, namely the *Adomian decomposition method* (ADM) and the *modified decomposition method*, are gaining popularity among scientists and engineers for solving highly nonlinear integral equations. Singular integral equations encountered by Abel can easily be solved by using the *Laplace transform method*. Volterra integral equations of convolution type can be solved using the Laplace transform method. Finally, for nonlinear problems, numerical techniques will be of extremely useful to solve the highly complicated problems.

It is to be noted that we will use two developed methods to solve many kinds of integral equations. **These two methods mostly give the solution in a series form or in an infinite geometric series**. It is therefore useful to present brief summaries of both series. More explanations of these two series can be found in calculus and algebra text books.

8.1 Taylor Series

In this section we will introduce a brief idea on ***Taylor series***. Recall that **the Taylor series exists for analytic functions only**.

Let $f(x)$ be a function that is ***infinitely differentiable*** in an interval $[b, c]$ that contains an ***interior point a***. The Taylor series of $f(x)$ generated at $x = a$ is given by the sigma notation

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \ldots\ldots\ldots\

$$+\frac{f^{(n)}(0)}{n!}(x)^n+\cdots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.108)$$

In what follows, we discuss few examples for the derivation of the Taylor series at $x = 0$.

Example 1. Find the Taylor series generated by $f(x) = \cos x$ at $x = 0$.

Solution: We list the function $f(x) = cosx$ its derivatives as follows:

$f^{(n)}(x)$	$f^{(n)}(0)$
$f(x) = cosx$	$f(0) = 1$
$f'(x) = -sinx$	$f'(0) = 0$
$f''(x) = -cosx$	$f''(0) = -1$
$f'''(x) = sinx$	$f'''(0) = 0$
$f^{(iv)}(x) = cosx$	$f^{(iv)}(0) = 1$
$\vdots$	

and so on. This gives the Taylor series for cos x by

$$cosx = 1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.109)$$

and in a compact form by

$$cosx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} \quad \forall x \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.110)$$

Similarly, we can easily show that

$$coshx = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.111)$$

and in a compact form by

$$coshx = \sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n} \quad \forall x \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.112)$$

Exercise. Find the Taylor series generated by $f(x) = \sin x$ at $x = 0$.

It was stated before that the proposed methods that will be used will give ***the solution in a series form***. The obtained series form may converge to a closed form solution, ***if such a solution exists***. ***Otherwise*** the truncated series solution may be used for numerical purposes. To get an exact solution, it is normal that we practice for the determination of this solution using Taylor series. ***In what follows, we study two examples only, and most of the integral equations will lead to these series***.

Example 2. Find the closed form function for the following series

$$f(x) = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \cdots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.113)$$

Solution: This series can be rewritten in the form

$$f(x) = 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \cdots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.114)$$

that will converge to the exact form

$$f(x) = e^{3x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.115)$$

Example 3. Find the closed form function for the following series

$$f(x) = 1 - x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \cdots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.116)$$

Solution: We group the series in the form

$$f(x) = \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) - \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right) \ldots\ldots\ldots\ldots\ldots\ldots (2.117)$$

that converges to

$$f(x) = cosx - sinx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.118)$$

8.2 Infinite Geometric Series

A *geometric series* is a series with a constant ratio between successive terms. The standard form of an infinite geometric series is given by

$$S_n = \sum_{k=0}^{n} a_1 r^k = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + \cdots + a_1 r^n \quad \ldots\ldots\ldots\ldots\ldots (2.119)$$

An *infinite geometric series* **converges** if and only if $|r| < 1$, otherwise it **diverges**. The sum of infinite geometric series, for $|r| < 1$, is given by

$$S_n = \frac{a_1}{1-r} \quad \ldots\ldots\ldots\ldots\

$$S = \frac{1}{1+\frac{1}{3}} = \frac{3}{4} \quad \ldots\ldots\ldots\

$$S = 1 + \frac{\pi}{8} sec^2 x \left(\frac{1}{1 - \frac{1}{2}} \right) = 1 + (2) \frac{\pi}{8} sec^2 x = 1 + \frac{\pi}{4} sec^2 x \quad \ldots\ldots\ldots\ldots\ldots (2.130)$$

Example 5. Simplify the following expression

$$S = x^4 + \frac{1}{14} x^2 + \frac{11}{70} x^2 + \frac{11}{350} x^2 + \frac{11}{1750} x^2 + \cdots \qquad \ldots\ldots\ldots\ldots\ldots (2.132)$$

Solution: The given expression can be rewritten as

$$S = x^4 + \frac{1}{14} x^2 + \frac{11}{70} x^2 \left(1 + \frac{1}{5} + \frac{1}{25} + \cdots \right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.133)$$

It is obvious that the third part is an infinite geometric series, with $a_1 = 1$ and $r = \frac{1}{5}$, $|r| < 1$. The sum is therefore given by

$$S = x^4 + \frac{1}{14} x^2 + \frac{11}{70} x^2 \left(\frac{1}{1 - \frac{1}{5}} \right) = x^4 + \frac{1}{14} x^2 + \frac{11}{70} x^2 \left(\frac{5}{4} \right) = x^4 + \frac{1}{14} x^2 + \frac{55}{280} x^2$$

$$= x^4 + \frac{1}{14} x^2 + \frac{11}{56} x^2 = x^4 + \frac{15}{56} x^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.134)$$

References

[1] J. H. He, Some Asymptotic Methods for Strongly Nonlinear Equations, International Journal of Modern Physics B, Vol. 20, No. 10 (2006) 1141–1199.

[2] R. P. Kanwal, Linear Integral Equations, Birkhauser, Boston, (1997).

Chapter 3

Volterra Integral Equations

3.1 Introduction

In the previous chapter, we have clearly defined the integral equations with some useful illustrations. This chapter deals with the Volterra integral equations and their solution techniques.
In this chapter we will be concerned with the *non-homogeneous Volterra integral equation* of the *second kind* of the form

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)\,dt \quad \ldots\ldots\ldots\

3.2 The method of successive approximations

In this method, we replace the unknown function $u(x)$ under the integral sign of the Volterra equation (3.1) by any selective real-valued continuous function $u_0(x)$, called the **zeroth approximation**. This substitution will give the first approximation $u_1(x)$ by

$$u_1(x) = f(x) + \lambda \int_0^x K(x,t)u_0(t)\,dt \quad \ldots\ldots\ldots\

$$u_1(x) = f(x) + \lambda \int_0^x K(x,t) f(t)\, dt$$

$$u_2(x) = f(x) + \lambda \int_0^x K(x,t) u_1(t)\, dt$$

$$u_3(x) = f(x) + \lambda \int_0^x K(x,t) u_2(t)\, dt$$

$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$

$$u_{n-2}(x) = f(x) + \lambda \int_0^x K(x,t) u_{n-1}(t)\, dt$$

$$u_n(x) = f(x) + \lambda \int_0^x K(x,t) u_{n-1}(t)\, dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.7)$$

The last equation is the recurrence relation. Consider

$$u_2(x) - u_1(x) = \lambda \int_0^x K(x,t) \left[f(t) + \lambda \int_0^t K(t,\tau) f(\tau) d\tau \right] dt$$

$$-\lambda \int_0^x K(x,t) f(t)\, dt$$

$$= \lambda^2 \int_0^x K(x,t) \int_0^t K(t,\tau) f(\tau) d\tau\, dt$$

$$= \lambda^2 \psi_2(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.8)$$

where

$$\psi_2(x) = \int_0^x K(x,t) \int_0^t K(t,\tau) f(\tau) d\tau\, dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.9)$$

Thus, it can be easily observed from equation (3.8) that

$$u_n(x) = \sum_{m=0}^{n} \lambda^m \psi_m \ldots\ldots\ldots\

$$\psi_m(x) = \int_0^x K_m(x,\tau) f(\tau) d\tau\,, \qquad m = 1,2,3,\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.12)$$

where the iterative kernels $K_1(x,t) \equiv K(x,t), K_2(x,t), K_3(x,t), \ldots$ are defined by the recurrence formula

$$K_{m+1}(x,t) = \int_t^x K(x,\tau) K_m(\tau,t) d\tau\,, \qquad m = 1,2,3,\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.13)$$

Thus, the solution for $u_n(x)$ can be written as

$$u_n(x) = f(x) + \sum_{m=1}^{n} \lambda^m \psi_m(x) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.14)$$

It is also plausible that we should be led to the solution of equation (3.1) by means of the sum **<u>if it exists</u>**, of the infinite series defined by equation (3.10). Thus, we have using equation (3.12)

$$u_n(x) = f(x) + \sum_{m=1}^{n} \lambda^m \psi_m(x) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.14)$$

$$u_n(x) = f(x) + \sum_{m=1}^{n} \lambda^m \int_0^x K_m(x,\tau) f(\tau) d\tau$$

$$u_n(x) = f(x) + \int_0^x \left\{ \sum_{m=1}^{n} \lambda^m \int_0^x K_m(x,\tau) \right\} f(\tau) d\tau \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.15)$$

hence it is also plausible that the solution of equation (3.1) will be given by as $n \to \infty$

$$\lim_{n\to\infty} u_n(x) = u(x)$$

$$= f(x) + \int_0^x \left\{ \sum_{m=1}^{n} \lambda^m \int_0^x K_m(x,\tau) \right\} f(\tau) d\tau$$

$$= f(x) + \lambda \int_0^x H(x,\tau;\lambda)\, f(\tau) d\tau \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.16)$$

where

$$H(x,\tau;\lambda) = \sum_{m=1}^{n} \lambda^m K_m(x,\tau) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\

and inverting this transform, we obtain

$$\boldsymbol{u(x) = \int_0^x \psi(x-t) f(t) dt} \quad \ldots\ldots\ldots\

where $\delta(x)$ is the Dirac delta function and we have used the integral property **[2]** to evaluate the integral. Because of the convolution type kernel, the result is amazingly simple.

(b) Solution by successive approximation

Let us assume that the zeroth approximation is

$$u_0(x) = 0 \quad \ldots\ldots (3.24)$$

Then the first approximation can be obtained as

$$u_1(x) = f(x) \quad \ldots\ldots (3.25)$$

Using this information in equation (3.22), the second approximation is given by

$$u_2(x) = f(x) + \lambda \int_0^x e^{x-t} f(t)dt \quad \ldots\ldots (3.26)$$

Proceeding in this manner, the third approximation can be obtained as

$$u_3(x) = f(x) + \lambda \int_0^x e^{x-t} u_2(t)dt$$

$$= f(x) + \lambda \int_0^x e^{x-t} \left\{ f(t) + \lambda \int_0^t e^{t-\tau} f(\tau)d\tau \right\} dt$$

$$= f(x) + \lambda \int_0^x e^{x-t} f(t)dt + \lambda^2 \int_0^x \int_0^t e^{x-\tau}\, d\tau dt$$

$$= f(x) + \lambda \int_0^x e^{x-t} f(t)dt + \lambda^2 \int_0^x (x-t)\, e^{x-t} f(t)dt$$

In the double integration the order of integration is changed to obtain the final result.

In a similar manner, **the fourth approximation** $u_4(x)$ can be at once written as

$$u_4(x) = f(x) + \lambda \int_0^x e^{x-t} f(t)dt + \lambda^2 \int_0^x (x-t)\, e^{x-t} f(t)dt + \lambda^3 \int_0^x \frac{(x-t)^2}{2!} e^{x-t} f(t)dt$$

Thus, continuing in this manner, we obtain as $n \to \infty$

$$u(x) = \lim_{n\to\infty} u_n(x)$$

$$= f(x) + \lambda \left\{ \int_0^x e^{x-t} \left(1 + \lambda(x-t) + \frac{1}{2!}\lambda^2(x-t)^2 + \cdots\right) f(t)dt \right\}$$

$$= f(x) + \lambda \int_0^x e^{x-t}.e^{\lambda(x-t)} f(t)dt$$

$$\boldsymbol{u(x) = f(x) + \lambda \int_0^x e^{(1+\lambda)(x-t)} f(t)dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{3.27})$$

which is the same as equation (3.23). Here, the resolvent kernel is $H(x,t;\lambda) = e^{(1+\lambda)(x-t)}$.

(c) Another method to determine the solution by the resolvent kernel

The procedure to determine the resolvent kernel is the following: Given that

$$u(x) = f(x) + \lambda \int_0^x e^{x-t}\, u(t)dt$$

Here, the kernel is $K(x,t) = e^{x-t}$. The solution by the ***successive approximation*** is

$$u(x) = f(x) + \lambda \int_0^x H(x,t;\lambda)\, f(t)dt$$

where the resolvent kernel is given by

$$H(x,t;\lambda) = \sum_{n=0} \lambda^n K_{n+1}(x,t)$$

in which

$$K_{n+1}(x,t) = \int_t^x K(x,\tau)K_n(\tau)d\tau$$

It is to be noted that $K_1(x,t) = K(x,t)$.

Thus, we obtain

$$K_2(x,t) = \int_t^x e^{x-\tau}e^{\tau-t}d\tau = e^{x-\tau}\int_t^x d\tau = (x-t)e^{x-t}$$

Similarly, proceeding in this manner, we obtain

$$K_3(x,t) = \int_t^x (e^{x-\tau})\big(e^{\tau-t}(\tau-t)\big)d\tau = e^{x-t}\frac{(x-t)^2}{2!}$$

$$K_4(x,t) = e^{x-t}\frac{(x-t)^3}{3!}$$

… … … … … …

$$K_{n+1}(x,t) = e^{x-t}\frac{(x-t)^n}{n!}$$

Hence the resolvent kernel is

$$H(x,t;\lambda) = \sum_{n=0} \lambda^n K_{n+1}(x,t)$$

$$= e^{x-t}\sum_{n=0}^{\infty}\frac{\big(\lambda(x-t)\big)^n}{n!} = e^{(1+\lambda)(x-t)}$$

Once the resolvent kernel is known the solution is obvious.

Example 3.2 : Solve the following linear Volterra integral equation

$$u(x) = x + \int_0^x (t-x)u(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.28)$$

Solution

(a) Solution by Laplace transform method

The Laplace transform of equation (3.28) yields

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{x\} - \mathcal{L}\{x\}\mathcal{L}\{u(x)\} = \frac{1}{s^2} - \frac{1}{s^2}\mathcal{L}\{u(x)\}$$

which reduces to $\mathcal{L}\{u(x)\} = \frac{1}{1+s^2}$ and its inverse solution is $\boldsymbol{u(x) = sinx}$. This is required solution.

(b) Solution by successive approximation

Let us set $u_0(x) = x$, then the first approximation is

$$u_1(x) = x + \int_0^x (t-x)\, tdt = x - \frac{x^3}{3!}$$

The second approximation can be calculated in the similar way that gives

$$u_2(x) = x + \int_0^x (t-x)\left(t - \frac{t^3}{6}\right)dt = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Proceeding in this way, we can obtain without much difficulty

$u_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. The final solution is

$$\boldsymbol{u(x)} = \lim_{n\to\infty} u_n(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = \boldsymbol{sinx}\ .$$

This is the same as before.

Chapter 3 (Cont.)

Remark: In ***Example 3.2*** (**pages 10-11**), we have solved the linear Volterra integral equation, Eq.(3.28), by using the method of **successive approximation**. To show that the final obtained solution $u(x) = sin(x)$ **does not depend on the selection of** $\boldsymbol{u_0(x)}$, we will solve the Eq.(3.28) by selecting

$$u_0(x) = 0. \tag{3.29}$$

Substituting Eq.(3.29) into Eq.(3.28) we find

$$u_1(x) = x + \int_0^x (t - x)\, u_0(t) dt, \tag{3.30}$$

and this gives the first approximation of $u(x)$ by

$$u_1(x) = x. \tag{3.31}$$

Substituting Eq.(3.31) into Eq.(3.30) we find

$$u_2(x) = x + \int_0^x (t - x)\, t dt, \tag{3.32}$$

where by integrating the right hand side of Eq.(3.32), the second approximation of $u(x)$

$$u_2(x) = x - \frac{1}{3!}x^3, \tag{3.33}$$

is readily obtained. Continuing in the same manner we find that the third approximation of $u(x)$ is

$$u_3(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5. \qquad (3.34)$$

Accordingly, the *n*th approximation is given by

$$u_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!}, \qquad n \geq 1. \qquad (3.35)$$

Consequently, the solution $u(x)$ of Eq.(3.28) is given by

$$u(x) = \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} \left(\sum_{k=1}^{n} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} \right), \qquad (3.36)$$

$$= sin(x).$$

the same answer we obtained above. This confirms the fact that the solution obtained does not depend on the selection of the zeroth approximation $u_0(x)$. ■

3.4 The Method of Successive Substitutions

This method introduces the solution of the integral equation in a series form through evaluating single integral and multiple integrals as well. The computational work needed in this method is huge compared with other techniques.

The Volterra integral equation of the second kind is rewritten here for ready reference,

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)\,dt. \qquad (3.37)$$

In this method, we set $x = t$ and $t = t_1$ in the Volterra integral equation (3.37), we obtain

$$u(t) = f(t) + \lambda \int_0^x K(t,t_1)u(t_1)\,dt_1. \qquad (3.38)$$

Replacing $u(t)$ in the right hand side of Eq.(3.37) by its obtained value given by Eq.(3.38), yields

$$u(x) = f(x) + \lambda \int_0^x K(x,t) \left[f(t) + \lambda \int_0^t K(t,t_1) u(t_1) dt_1 \right] dt, \qquad (3.39)$$

$$= f(x) + \lambda \int_0^x K(x,t)\, f(t) dt + \lambda^2 \int_0^x K(x,t) \int_0^t K(t,t_1) u(t_1) dt_1\, dt. \qquad (3.40)$$

Substituting $x = t_1$ and $t = t_2$ in Eq.(3.37) we obtain

$$u(t_1) = f(t_1) + \lambda \int_0^x K(t_1,t_2) u(t_2)\, dt_2. \qquad (3.41)$$

Substituting the value of $u(t_1)$ obtained in Eq.(3.41) into the right hand side of Eq.(3.40) leads to

$$u(x) = f(x) + \lambda \int_0^x K(x,t)\, f(t) dt$$

$$+\lambda^2 \int_0^x \int_0^t K(x,t) K(t,t_1) f(t_1) dt_1 dt$$

$$+\lambda^3 \int_0^x \int_0^t \int_0^{t^1} K(x,t) K(t,t_1)\, K(t_1,t_2) u(t_2) dt_2 dt_1 dt$$

$$+ \cdots \qquad (3.42)$$

Accordingly, the general series form for $u(x)$ can be rewritten as

$$u(x) = f(x) + \lambda \int_0^x K(x,t)\, f(t)dt + \lambda^2 \int_0^x K(x,t) \int_0^t K(t,t_1) f(t_1) dt_1\, dt + \cdots$$
$$+ \lambda^n \int_0^x K(x,t) \int_0^t K(t,t_1) \ldots \times \int_0^{t_{n-2}} K(t_{n-2},t_{n-1}) f(t_{n-1}) dt_{n-1} \ldots dt_1 dt$$
$$+ R_{n+1}(x), \qquad (3.43)$$

where

$$R_{n+1}(x) = \lambda^{n+1} \int_0^x K(x,t) \int_0^t K(t,t_1) \ldots \int_0^{t_{n-1}} K(t_{n-1},t_n) u(t_n) dt_n \ldots dt_1 dt \ \ldots \ (3.44)$$

is the remainder after *n* terms. It can be shown that **(see Ref. [1])** $\lim_{n\to\infty} R_{n+1} = 0$.

It is to be noted here that in this method the unknown function $u(x)$ is substituted by the given function $f(x)$ that makes the evaluation of the multiple integrals easily computable. This substitution of $u(x)$ occurs several times through the integrals and **hence this is why it is called the method of successive substitutions**. The technique will be illustrated by discussing the following examples.

Theorem 3.1: If

(a) $u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)\, dt, a\ is\ a\ constant \qquad (3.44a)$

(b) $K(x,t)$ is real and continuous in the rectangular R, for which $a \le x \le b, a \le t \le b$, $|K(x,t)| \le M\ in\ R, \qquad K(x,t) \ne 0.$

(c) $f(x) \ne 0$, is real and continuous in $I : a \le x \le b$.

(d) λ is a constant.

Then the Eq. (3.37) has one and only one continuous solution $u(x)$ in I, and this solution is given by the absolutely and uniformly convergent series (3.42).

The results of this theorem hold without change for the equation

$$u(x) = f(x) + \int_0^x K(x,t)u(t)\,dt, \qquad for\ \lambda = 1.$$

Example 1: Solve the following Volterra integral equation

$$u(x) = x - \int_0^x (x-t)u(t)\,dt, \tag{3.45}$$

by using the method of successive substitutions**.**

Solution: Substituting $\lambda = -1$, $f(x) = x$, and $K(x,t) = (x-t)$ into Eq.(3.42) yields

$$u(x) = x - \int_0^x (x-t)\,tdt + \int_0^x \int_0^t (x-t)(x-t_1)t_1 dt_1 dt + \cdots, \tag{3.46}$$

or equivalently

$$u(x) = x - \int_0^x (xt - t^2)\,dt + \int_0^x \int_0^t (x-t)(tt_1 - t_1^2)dt_1 dt + \cdots \tag{3.47}$$

Therefore we obtain the solution in a series form

$$u(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots \tag{3.48}$$

or in a closed form

$$u(x) = sin(x), \tag{3.49}$$

upon using the Taylor expansion for sin *x*. ■

Example 2: Solve the following Volterra integral equation

$$u(x) = x^2 - x^4 + \int_0^x 4tu(t)\,dt, \tag{3.50}$$

by using the method of successive substitutions**.**

Solution: Substituting $\lambda = 4$, $f(x) = x^2 - x^4$, and $K(x,t) = t$ into Eq.(3.42) yields

$$u(x) = x^2 - x^4 + 4\int_0^x t(t^2 - t^4)\,dt + 16\int_0^x \int_0^t tt_1(t_1^2 - t_1^4)dt_1 dt$$

$$+64\int_0^x \int_0^t \int_0^{t_1} t\,t_1 t_2(t_2^2 - t_2^4)\,dt_2 dt_1 dt + \cdots, \qquad (3.51)$$

and this will yield

$$u(x) = x^2 - x^4 + x^4 - \frac{2}{3}x^6 + \frac{2}{3}x^6 - \frac{1}{3}x^8 + \frac{1}{3}x^8 + \cdots \qquad (3.52)$$

Consequently we easily obtain the exact solution

$$u(x) = x^2, \qquad (3.53)$$

upon cancelling the similar terms with opposite signs in Eq.(3.52). ■

3.5 The Series Solution Method

In this section we will introduce a practical method, called the ***series solution method***, to solve the Volterra integral equation with variable limits of integration

$$u(x) = f(x) + \lambda\int_0^x K(x,t)u(t)\,dt. \qquad (3.54)$$

We will follow a parallel approach to the method of the series solution around an ordinary point that usually applied in solving an ordinary differential equation. The method is applicable provided that $u(x)$ is an analytic function, i.e. $u(x)$ has a Taylor expansion at *x* = 0. Accordingly, $u(x)$ can be expressed by a series expansion given by

$$u(x) = \sum_{n=0}^{\infty} a_n\,x^n, \qquad (3.55)$$

where the coefficients a_n are constants that will be determined. Substituting Eq.(3.55) into both sides of Eq.(3.54) yields

$$\sum_{n=0}^{\infty} a_n\, x^n = f(x) + \lambda \int_0^x K(x,t)\left(\sum_{n=0}^{\infty} a_n\, t^n\right) dt\,, \tag{3.56}$$

so that by using few terms of the expansions in both sides, we find

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = f(x) &+ \lambda \int_0^x K(x,t)\, a_0 dt \\ &+ \lambda \int_0^x K(x,t)\, a_1 t dt \\ &+ \lambda \int_0^x K(x,t)\, a_2 t^2 dt \\ &+ \lambda \int_0^x K(x,t)\, a_3 t^3 dt \\ &+ \cdots \end{aligned} \tag{3.57}$$

In view of Eq.(3.57), the integral equation Eq.(3.54) will be reduced to several traditional integrals, with defined integrands having terms of the form $t^n, n \geq 0$ only. We then write the Taylor expansions for $f(x)$ and evaluate the first few integrals in Eq.(3.57). Having performed the evaluation, we equate the coefficients of like powers of *x* in both sides of Eq. (3.57). This will lead to a complete determination of the coefficients $a_0, a_1, a_3, \ldots$ Consequently, substituting the obtained coefficients $a_n, n \geq 0$, in Eq.(3.55) produces the solution in a series form. This may lead to a solution in a closed form if the expansion obtained is a Taylor expansion to a well known elementary function.

It seems reasonable to illustrate the series solution method by discussing the following examples.

Example 1: Solve the following Volterra integral equation

$$u(x) = 1 + \int_0^x (t - x)u(t)\, dt, \tag{3.58}$$

by using the series solution method.

Solution: Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n\, x^n\,, \tag{3.59}$$

into both sides of the equation Eq.(3.58) leads to

$$\sum_{n=0}^{\infty} a_n\, x^n = 1 + \int_0^x (t - x)\left(\sum_{n=0}^{\infty} a_n\, t^n\right) dt, \tag{3.60}$$

which gives

$$\sum_{n=0}^{\infty} a_n\, x^n = 1 + \int_0^x \left(\sum_{n=0}^{\infty} a_n\, t^{n+1} - x\sum_{n=0}^{\infty} a_n\, t^n\right) dt\,. \tag{3.61}$$

Evaluating the regular integrals at the right hand side that involve terms of the form $t^n, n \geq 0$ yields

$$\sum_{n=0}^{\infty} a_n\, x^n = 1 - \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}\, a_n x^{n+2}\,, \tag{3.62}$$

or equivalently

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 - \frac{1}{2!} a_0 x^2 - \frac{1}{3!} a_1 x^3 - \frac{1}{12} a_2 x^4 + \cdots \tag{3.63}$$

Equating the coefficients of like powers of *x* in both sides we find

$$a_0 = 1,$$
$$a_1 = 0,$$

$$a_2 = -\frac{1}{2!}, \tag{3.64}$$
$$a_3 = 0,$$
$$a_4 = \frac{1}{4!},$$

and generally

$$a_{2n} = (-1)^n \frac{1}{(2n)!}, \quad for\ n \geq 0 \tag{3.65}$$

$$a_{2n+1} = 0 \quad for\ n \geq 0 \tag{3.66}$$

Using Eq.(3.59) we find the solution in a series form

$$u(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \tag{3.67}$$

and in a closed form by

$$u(x) = cos(x). \quad \blacksquare \tag{3.68}$$

Example 2: Solve the following integral equation

$$u(x) = 2\ cosh(x) - x\ sinh(x) - 1 + \int_0^x tu(t)\,dt, \tag{3.69}$$

by using the series solution method.

Solution: Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n\, x^n\,, \tag{3.70}$$

into both sides of the equation Eq.(3.69) leads to

$$\sum_{n=0}^{\infty} a_n\, x^n = 2\left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right) - x\left(\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}\right) - 1 + \int_0^x t\left(\sum_{n=0}^{\infty} a_n\, t^n\right) dt, \tag{371}$$

where we use the Taylor series of $cosh(x)$ and $sinh(x)$. Using a few terms from each series involved and integrating the resulting integrals at the right hand side we obtain

$$a_0 + a_1 x + a_2 x^2 + \cdots = 2\left(1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) - x\left(x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right)$$

$$-1 + \frac{1}{2}a_0 x^2 + \frac{1}{3}a_1 x^3 + \frac{1}{4}a_2 x^4 + \cdots \qquad (3.72)$$

Equating the coefficients of like powers of *x* in in both sides of Eq.(3.72) yields

$$\begin{aligned} a_0 &= 1, \\ a_1 &= 0, \\ a_2 &= -\frac{1}{2!}, \\ a_3 &= 0, \\ a_4 &= \frac{1}{4!}, \end{aligned} \qquad (3.73)$$

and generally

$$a_{2n} = (-1)^n \frac{1}{(2n)!}, \quad for\ n \geq 0 \qquad (3.74)$$

$$a_{2n+1} = 0 \quad for\ n \geq 0 \qquad (3.75)$$

Using Eq.(3.70) we find the solution in a series form

$$u(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \cdots \qquad (3.76)$$

and in a closed form by

$$u(x) = cosh(x). \qquad (3.77)$$

obtained by using the Taylor expansion of $cosh(x)$. ■

Remark: In **chapter 2**, (sec.5, page 10) the process of converting Volterra integral equations to equivalent initial value problems (IVP) has been discussed in detail. There is a variety on analytic and numerical techniques, traditional and new, that are usually used in studying Volterra integral equations. In the following we will discuss the recent methods that handle successfully the linear Volterra integral equations.

3.6 The Adomian Decomposition Method

Adomian **[2]** developed the so-called *Adomian decomposition method* or simply the *decomposition method* (ADM). The Adomian decomposition method appears to work for *linear* and *nonlinear differential equations*, *integral equations*, and *integro-differential equations*. The method was introduced by Adomian in early 1990 in his books **[2]** and **[3]** and other related research papers **[4]** and **[5]**. The method essentially is a power series method similar to the perturbation technique. That is, this method provides the solution in a series form as will be seen later.

The method was applied mostly to *ordinary* and *partial differential equations*, and was *rarely used for integral equations* in **[2]** and **[3]**. The concept of convergence of the solution obtained by this method was addressed extensively in the literature.

In the decomposition method we usually express the solution $u(x)$ of the Volterra integral equation

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)\,dt, \tag{3.78}$$

in a series form defined by

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{3.79}$$

Substituting the decomposition Eq.(3.79) into both sides of Eq.(3.78) yields

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x K(x,t)\left\{\sum_{n=0}^{\infty} u_n(t)\right\}dt, \tag{3.80}$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = f(x) + \lambda \int_0^x K(x,t)\,u_0(t)dt + \lambda \int_0^x K(x,t)\,u_1(t)dt \tag{3.81}$$

$$+\lambda \int_0^x K(x,t)\, u_2(t)dt$$
$$+\cdots$$

The components $u_0(x), u_1(x), u_2(x), \ldots$ of the unknown function $u(x)$ can be **completely** determined in a recurrence manner if we set

$$u_0(x) = f(x), \tag{3.82}$$

$$u_1(x) = \lambda \int_0^x K(x,t)\, u_0(t)dt, \tag{3.83}$$

$$u_2(x) = \lambda \int_0^x K(x,t)\, u_1(t)dt, \tag{3.84}$$

and so on. The above discussed scheme for the determination of the components $u_0(x), u_1(x), u_2(x), \ldots$ of the solution $u(x)$ of Eq.(3.78) can be written in a recursive manner by

$$\boldsymbol{u_n(x) = \lambda \int_0^x K(x,t)\, u_{n-1}(t)dt.} \tag{3.85}$$

and so on. This set of equations **(3.85)** can be written in compact recurrence scheme as

$$\boldsymbol{u_0(x) = f(x),} \tag{3.86}$$

$$\boldsymbol{u_{n+1}(x) = \lambda \int_0^x K(x,t)\, u_n(t)dt\ , \qquad n \geq 0.} \tag{3.87}$$

In view of Eq.(3.86) and Eq.(3.87), the components $u_0(x), u_1(x), u_2(x), \ldots$ follow immediately. With these components determined, the solution $u(x)$ of Eq.(3.78) is readily determined in a series form using the decomposition Eq.(3.79). It is important to note that the obtained series for $u(x)$ converges to the exact solution in a closed form **if such solution exists** as will be seen later.

However, for concrete problems, where exact solution cannot be evaluated, i.e. it may not be possible to integrate the kernel for many components; a truncated series $\sum_{n=0}^{k} u_n(x)$ is usually used to approximate the solution $u(x)$ and this can be used for numerical purposes. We point out here that few terms of the truncated series usually provide the higher accuracy level of the approximate solution if compared with the existing numerical techniques.

There is another point that needs to be addressed; that is the convergence of the solution of the infinite series. This problem was addressed by many previous workers in this area (see Refs. **[6]** and **[7]**).

In general, the decomposition technique proved to be effective and reliable even if applied to nonlinear Volterra integral equations as will be discussed in the forthcoming Chapters.

The following illustrative examples will be discussed to explain the above outlined decomposition method.

Example 1: Solve the Volterra integral equation

$$u(x) = 1 + \int_0^x u(t)\,dt, \tag{3.88}$$

by the decomposition method.

Solution: It is clear that $f(x) = 1,\ \lambda = 1,\ K(x,t) = 1$. Using the decomposition series solution Eq.(3.79) and the recursive scheme Eq.(3.86) and Eq.(3.87) to determine the components u_n , $n \geq 0$, we find

$$u_0(x) = 1 \tag{3.89}$$

$$u_1(x) = \int_0^x u_0(t)\,dt = \int_0^x dt\ = x \tag{3.90}$$

$$u_2(x) = \int_0^x u_1(t)\,dt\ = \int_0^x t\,dt\ = \frac{1}{2!}x^2 \tag{3.91}$$

and so on. Noting that

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots, \tag{3.92}$$

we can easily obtain the solution in a series form given by

$$u(x) = 1 + x + \frac{1}{2!}x^2 + \cdots \tag{3.93}$$

and this converges to the closed form solution

$$u(x) = e^x\,, \tag{3.94}$$

obtained upon using the Taylor expansion for e^x. ■

Example 2: Solve the Volterra integral equation

$$u(x) = x + \int_0^x (t - x)u(t)dt, \tag{3.95}$$

by the decomposition method.

Solution: Proceeding as in **Example 1**, we set

Consider the solution in the series form $u(x) = \sum_{n=0}^{\infty} u_n(x)$. Then substituting this series into the given equation, we have

$$\sum_{n=0}^{\infty} u_n(x) = x + \int_0^x (t - x) \sum_{n=0}^{\infty} u_n(t)\, dt. \tag{3.96}$$

Now decomposing the different terms in the following manner, we get a set of solutions

$$u_0(x) = x, \tag{3.97}$$

$$u_1(x) = \int_0^x (t - x)u_0(t)dt = \int_0^x (t - x)tdt = \frac{-x^3}{3!}, \tag{3.98}$$

$$u_2(x) = \int_0^x (t - x)u_1(t)dt = \int_0^x (t - x)\frac{-t^3}{3!}\,dt = \frac{x^5}{5!}, \tag{3.99}$$

Consequently, the solution of Eq.(3.95) in a series form is given by

$$u(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \tag{3.100}$$

and in a closed form by

$$u(x) = sin(x)\,, \tag{3.101}$$

obtained by using the Taylor expansion of $sin(x)$. ■

Note that; by the ratio test where $u_n(x) = (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, it can be easily shown that $|x| < \infty$, that means the series is **convergent** for all values of *x* in a finite domain. By taking the Laplace transform of the integral equation, it is easy to see that $\mathcal{L}\{u(x)\} = \frac{1}{s^2+1}$ and so $\boldsymbol{u(x) = sinx}$.

It was mentioned above that it is **not always possible** to determine a solution in a closed form, but instead the solution obtained may be expressed in a series form. The difference between the closed form solution and the power series solution has been illustrated before in previous sections. However the series solution is usually employed for **numerical approximations**, and the more terms we obtain provide more accuracy in the approximation of the solution. In the next example we discuss again the solution expressed in a series form.

Example 3: Solve the Volterra integral equation

$$u(x) = 1 + \frac{1}{2}\int_0^x xt^2 u(t)dt, \tag{3.102}$$

by the decomposition method.

Solution: Following the procedure used above, we find

$$u_0(x) = 1, \tag{3.103}$$

$$u_1(x) = \frac{1}{2}x\int_0^x t^2dt = \frac{1}{6}x^4\,, \tag{3.104}$$

$$u_2(x) = \frac{1}{12}x\int_0^x t^6dt = \frac{1}{84}x^8\,, \tag{3.105}$$

$$u_3(x) = \frac{1}{168}x\int_0^x t^{10}dt = \frac{1}{1848}x^{12}\,, \tag{3.106}$$

Consequently, the solution of Eq.(3.102) in a series form is given by

$$u(x) = 1 + \frac{1}{6}x^4 + \frac{1}{84}x^8 + \frac{1}{1848}x^{12} + \cdots, \tag{3.107}$$

so that a closed form for $u(x)$ does not appear obtainable. ■

Even though the decomposition method proved to be powerful and reliable, but it can be used sometimes in a more effective manner which we called the ***modified decomposition method***. The volume of calculations will be reduced by evaluating only the first two components $u_0(x)$ and $u_1(x)$. The modified technique works for specific problems where the function $f(x)$ in Eq.(3.78) consists of at least two terms.

► 3.6.1 The Modified Decomposition Method

It is worth noting that the Adomian decomposition method may be sometimes implemented in a different but easier manner in order to facilitate the computational work. It is recommended to apply the modified decomposition method, for cases where the nonhomogeneous part $f(x)$ in Eq.(3.78) consists of a combination of many terms.

In Volterra integral equations where the nonhomogeneous part $f(x)$ in Eq.(3.78) consists of a polynomial that includes many terms, or in the case $f(x)$ contains a combination of polynomial and other trigonometric or transcendental functions, the modified decomposition method showed to work extremely well. This modified technique, as will be seen later, will minimize the volume of calculations and reduce the several integral evaluations that result in applying the standard Adomian decomposition method.

It is also of interest, before giving a clear discussion of this method, to note that this modified technique will be carried out with promising results in Fredholm integral equations and nonlinear integral equations in forthcoming chapters. The technique avoids the cumbersome integrations of other methods.

In the modified method, we split the given function $f(x)$ into two parts defined by

$$f(x) = f_0(x) + f_1(x), \tag{3.108}$$

where $f_0(x)$ consists of one term only, or if needed of more terms in fewer other cases, of $f(x)$, and $f_1(x)$ includes the remaining terms of $f(x)$. We note that a necessary condition is required to apply this approach in that $f(x)$ should consist of more than one term as shown by Eq.(3.108). In view of Eq.(3.108), the integral equation Eq.(3.78) becomes

$$u(x) = f_0(x) + f_1(x) + \lambda \int_0^x K(x,t)u(t)\,dt. \tag{3.109}$$

Substituting the decomposition given by Eq.(3.79) into Eq.(3.109) and using few terms of the expansions we obtain

$$\begin{aligned} u_0(x) + u_1(x) + u_2(x) + \cdots = f_0(x) + f_1(x) &+ \lambda \int_0^x K(x,t)\,u_0(t)dt \\ &+\lambda \int_0^x K(x,t)\,u_1(t)dt \\ &+\lambda \int_0^x K(x,t)\,u_2(t)dt \\ &+\lambda \int_0^x K(x,t)\,u_3(t)dt \\ &+\cdots \end{aligned} \tag{3.110}$$

Consequently, the components $u_i(x)$, $i \geq 0$ of the unknown function $u(x)$ can be completely determined in a modified recurrence relation if we assign $f_0(x)$ only to the component $u_0(x)$, whereas the component $f_1(x)$ will be added to the formula of the component $u_1(x)$ given in Eq.(3.83). In other words, the modified recurrence relation

$$u_0(x) = f_0(x), \tag{3.111}$$

$$u_1(x) = f_1(x) + \lambda \int_0^x K(x,t)\, u_0(t)dt, \tag{3.112}$$

$$u_2(x) = \lambda \int_0^x K(x,t)\, u_1(t)dt, \tag{3.113}$$

$$u_3(x) = \lambda \int_0^x K(x,t)\, u_2(t)dt, \tag{3.114}$$

and so on. The method discussed above for the determination of the components of the solution $u(x)$ of Eq.(3.78) can be written in a recursive relationship by

$$u_0(x) = f_0(x), \tag{3.115}$$

$$u_1(x) = f_1(x) + \lambda \int_0^x K(x,t)\, u_0(t)dt, \tag{3.116}$$

$$u_{n+1}(x) = \lambda \int_0^x K(x,t)\, u_n(t)dt\,, \;\; n \geq 1. \tag{3.117}$$

In most problems, we need to use Eq.(3.115) and Eq.(3.116) only. For illustration purposes, we study the following examples.

Example 4: Solve the Volterra integral equation

$$u(x) = sec(x)tan(x) - \frac{1}{4}\left(e^{sec(x)} - e\right)x + \frac{1}{4}\int_0^x x\, e^{sec(t)} u(t)dt, \qquad x < \frac{\pi}{2}, \tag{3.118}$$

by using the modified decomposition method.

Solution: Using the modified decomposition method as discussed above, we first decompose the function $f(x)$ into

$$f_0(x) = sec(x)tan(x), \tag{3.119}$$

and

$$f_1(x) = -\frac{1}{4}\left(e^{sec(x)} - e\right)x, \tag{3.120}$$

Consequently, we find

$$u_0(x) = sec(x)tan(x), \tag{3.121}$$

$$u_1(x) = -\frac{1}{4}\left(e^{sec(x)} - e\right)x + \frac{1}{4}\int_0^x x\, e^{sec(t)} u_0(t)dt, \tag{3.122}$$

$$= -\frac{1}{4}\left(e^{sec(x)} - e\right)x + \frac{1}{4}\int_0^x sec(t)\, tan(t) e^{sec(t)} dt = 0, \tag{3.123}$$

obtained by integrating by substitution where we set $y = sec(t)$. Accordingly, other components $u_i(x) = 0, for\ i \geq 2$. Therefore, the exact solution

$$u(x) = sec(x)tan(x), \tag{3.124}$$

follows immediately. It is clear that two components are calculated to determine the exact solution. modified recurrence relation. ■

Example 5: Solve the Volterra integral equation

$$u(x) = cos(x) + sin(x) - \int_0^x u(t)\, dt, \tag{3.125}$$

by using the modified decomposition method.

Solution: Using the modified decomposition method as discussed above, we first decompose the function $f(x)$ into

$$f_0(x) = cos(x), \tag{3.126}$$

and

$$f_1(x) = sin(x), \tag{3.127}$$

Consequently, we find

$$u_0(x) = cos(x), \tag{3.128}$$

$$u_1(x) = -\int_0^x u_0(t)\,dt = 0. \tag{3.129}$$

Accordingly, other components $u_i(x) = 0, for\ i \geq 2$. Therefore, the exact solution

$$u(x) = cos(x), \tag{3.130}$$

follows immediately. ■

► 3.6.2 The Noise Term Phenomenon

The noise terms phenomenon is a method which accelerates the convergence of the Adomian decomposition method. The noise terms phenomenon demonstrates a useful tool for fast convergence of the solution. It was proved that the noise terms phenomenon may appear only for nonhomogeneous PDEs of any order or nonhomogeneous integral equations of any kind. The noise terms, if exist, it will appear in all components. The noise terms that may exist in the components $u_0(x)$ and $u_1(x)$ will give the solution in a closed form by using the components $u_0(x)$ and $u_1(x)$.

The noise terms are the identical terms with opposite signs that arise in the components, and may exist only for nonhomogeneous equations. It was found that by canceling the noise terms between $u_0(x)$ and $u_1(x)$, even though $u_1(x)$ contains other terms, the remaining non-canceled terms of $u_0(x)$ may give the exact solution of the equation.
Notice that it is necessary to verify that the non-canceled terms of $u_0(x)$ satisfy the PDE or the integral equation. In case the non-canceled terms of $u_0(x)$ did not satisfy the given equation, or the noise terms did not appear, then we proceed to determine more components of the series solution. **That is, the appearance of the noise terms between $u_0(x)$ and $u_1(x)$ is not always enough to give the exact solution**, hence it is necessary to show that the non-canceled terms of $u_0(x)$ satisfy the given integral equation.

We point out that noise terms **may appear only for nonhomogeneous problems**, whereas **homogeneous problems do not give rise to noise terms**. Moreover, it was proved that the appearance of the noise terms is governed by a necessary condition, in that the zeroth component $u_0(x)$ must contain the exact solution $u(x)$ among other terms.

In conclusion:

It was shown by many authors that non homogeneity condition is **not sufficient** to give noise terms. Moreover, it is necessary that the zeroth component $u_0(x)$ should contain the exact solution $u(x)$ among other terms, and this only may give noise terms about components. Moreover, noise terms may appear if the exact solution $u(x)$ is part of the zeroth component $u_0(x)$. The phenomenon of the useful noise terms will be explained by the following illustrative examples.

Example 6: Solve the Volterra integral equation by using noise terms phenomenon

$$u(x) = 6x + 3x^2 - \int_0^x u(t)\,dt, \tag{3.131}$$

Solution: The Adomian method admits the use of the recurrence relation

$$u_0(x) = 6x + 3x^2, \tag{3.132}$$

$$u_{k+1}(x) = -\int_0^x t u_k(t) dt\,, \qquad k \geq 0. \tag{3.133}$$

This gives

$$u_0(x) = 6x + 3x^2, \tag{3.134}$$

$$u_1(x) = -\int_0^x t u_0(t) dt = -3x^2 - x^3\,. \tag{3.135}$$

The noise terms $\mp 3x^2$ appear in $u_0(x)$ and $u_1(x)$. Canceling this term from the zeroth component $u_0(x)$ gives the exact solution

$$u(x) = 6x, \tag{3.136}$$

that satisfies the integral equation. This should be justified by substitution. ■

Example 7: Solve the Volterra integral equation by using noise terms phenomenon

$$u(x) = sinh(x) + x\ sinh(x) - x^2 cosh(x) + \int_0^x xtu(t)\,dt, \qquad (3.137)$$

Solution: Following the standard Adomian method we set the recurrence relation

$$u_0(x) = sinh(x) + x\ sinh(x) - x^2 cosh(x), \qquad (3.138)$$

$$u_{k+1}(x) = \int_0^x xtu_k(t)dt. \qquad (3.139)$$

This gives

$$u_0(x) = sinh(x) + x\ sinh(x) - x^2 cosh(x), \qquad (3.140)$$

$$u_1(x) = \int_0^x tu_0(t)dt = -x\ sinh(x) + x^2\ cosh(x) + other\ terms\,. \qquad (3.141)$$

The noise terms $\mp x\ sinh(x)$ and $\mp x^2\ cosh(x)$ appear in $u_0(x)$ and $u_1(x)$. Canceling these terms from the zeroth component $u_0(x)$ gives the exact solution

$$u(x) = sinh(x), \qquad (3.142)$$

that satisfies the integral equation, that should be justified. The other terms of $u_1(x)$ vanish in the limit with other terms of the other components. ■

3.7 The Variational Iteration Method (VIM)

In this section we will present the ***variational iteration method*** (VIM) that was used recently in the literature to handle both differential and integral equations, linear and nonlinear. Recall that the Adomian decomposition method gives the components of the

decomposition series by using a recurrence relation. **Unlike** the ADM, the (VIM) gives successive approximations of the solution that may converge rapidly to the exact solution if such a solution exists. However, for concrete problems the obtained approximations can be used for numerical reasons.
Ji-Huan He **[9–11]** developed this method, that proved to be reliable in the study of linear and nonlinear, and homogeneous and inhomogeneous equations.

It is interesting to note that the (VIM) is used for a differential equation, ordinary or partial, and for an integro-differential equation. This means that to use this method for solving integral equations, we first should convert the integral equation to its differential equation, or to its equivalent integro-differential equation, by using any appropriate method. **Unlike** the Adomian method that can be used directly to solve an integral equation, the VIM will be employed to the converted differential or integro differential equation.

One significant feature of this method is that it can handle especially nonlinear problems without the use of the so-called Adomian polynomials as required by the Adomian decomposition method.

In this section, we will apply the (VIM) to handle Volterra integral equation. The method works effectively if the kernel $K(x, t)$ is **separable** of the form $K(x,t) = g(x)h(t)$. The Volterra integral equation can be converted to an identical Volterra integro differential equation by differentiating both sides, where an initial condition should also be derived. For simplicity, we will study only the cases where $g(x) = x^n$, $n \geq 1$.
In what follows we will present the main steps for using this method.

The standard Volterra integral equation is of the form

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)\,dt, \qquad (3.143)$$

or equivalently

$$u(x) = f(x) + g(x) \int_0^x h(t)u(t)\,dt\,, \qquad K(x,t) = g(x)h(t) \qquad (3.144)$$

Recall that the integral at the right side of Eq.(3.144) depends on t only, hence it is equivalent to a constant. Differentiating both sides of Eq.(3.144) with respect to x gives

$$u'(x) = f'(x) + g'(x)\int_0^x h(t)u(t)\,dt\,, \qquad (3.145)$$

The (VIM) admits the use of a correction functional for the integro-differential equation Eq.(3.145) in the form

$$u_{n+1}(x) = u_n(x) + \int_0^x \gamma(\xi)\left(u_n'(\xi) - f'(\xi) - g'(\xi)\int_0^x h(r)\tilde{u}_n(r)\,dr\right)d\xi \quad (3.146)$$

where γ is a general **Lagrange multiplier**. Note that the Lagrange multiplier γ may be a **constant** or a **function**, and $\tilde{u}_n$ is a restricted value that means it behaves as a constant, hence $\delta\tilde{u}_n = 0$, where δ is the **variational derivative**. The Lagrange multiplier γ can be identified optimally via the variational theory as proved in the literature.

The (VIM) depends mainly on two essential steps. We first should determine the Lagrange multiplier $\gamma(\xi)$ that can be identified optimally via the variational theory where integration by parts should be used. As stated before, γ may be a **constant** or a **function**, and **it is different from one question to another**. A list of some of these Lagrange multipliers will be given later, but this list does not cover all differential or integral problems. However for our present use in this chapter, $\gamma(\xi) = -1$ for first order integro-differential equations. Having determined γ, an iteration formula, restricted variation, given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - f'(\xi) - g'(\xi)\int_0^x h(r)u_n(r)\,dr\right)d\xi \qquad (3.147)$$

is used to determine the successive approximations $u_{n+1}(x)$, $n \geq 0$ of the solution $u(x)$.

Notice that $u_n(x)$ gives the successive approximations of the solution and not the components as in the case when Adomian method is used. In other words, the correction functional will give several approximations of the solution.

The zeroth approximation $u_0(x)$ can be any selective function. However, using the given initial value $u(0)$ is preferably used for the selective zeroth approximation $u_0(x)$ as will be seen later. Consequently, the solution is given by

$$u(x) = \lim_{n\to\infty} u_n(x). \qquad (3.148)$$

The determination of the Lagrange multiplier is essential for the use of the correction functional. In what follows, we summarize some iteration formulae that show ODE, its corresponding Lagrange multipliers for this kind of ODEs, and its correction functional respectively **[12]**:

i. $$\begin{cases} u' + f\big(u(\xi), u'(\xi)\big) = 0\,, & \gamma = -1 \\ u_{n+1} = u_n - \int_0^x [u_n' + f(u_n, u_n')]d\xi\,, \end{cases}$$

ii. $$\begin{cases} u'' + f\big(u(\xi), u'(\xi), u''(\xi)\big) = 0\,, & \gamma = (\xi - x) \\ u_{n+1} = u_n + \int_0^x (\xi - x)[u_n'' + f(u_n, u_n', u_n'')]d\xi\,, \end{cases}$$

iii. $$\begin{cases} u''' + f\big(u(\xi), u'(\xi), u''(\xi), u'''(\xi)\big) = 0\,, & \gamma = -\frac{1}{2!}(\xi - x)^2 \\ u_{n+1} = u_n - \int_0^x \frac{1}{2!}(\xi - x)^2[u_n''' + f(u_n, u_n', u_n'', u_n''')]d\xi\,, \end{cases}$$

iv. $$\begin{cases} u^{(iv)} + f\left(u(\xi), u'(\xi), u''(\xi), u'''(\xi), u^{(iv)}(\xi)\right) = 0\,, & \gamma = \frac{1}{3!}(\xi - x)^3 \\ u_{n+1} = u_n + \int_0^x \frac{1}{3!}(\xi - x)^3 \left[u_n^{(iv)} + f\left(u_n, u_n', u_n'', u_n''', u_n^{(iv)}\right)\right] d\xi\,, \end{cases}$$

and generally

v. $$\begin{cases} u^{(n)} + f\left(u(\xi), u'(\xi), u''(\xi), \dots, u^{(n)}(\xi)\right) = 0\,, \ \gamma = (-1)^n \frac{1}{(n-1)!}(\xi - x)^{(n-1)} \\ u_{n+1} = u_n + (-1)^n \int_0^x \frac{1}{(n-1)!}(\xi - x)^{(n-1)} \left[u_n^{(n)} + f\left(u_n, u_n', u_n'', \dots, u_n^{(n)}\right)\right] d\xi\,, \end{cases}$$

for $n \geq 1$.

The (VIM) admits the use of a correction functional in the form

$$u_{n+1}(x) = u_n(x) + \int_0^x \gamma(\xi)\left(Lu_n(\xi) + N\tilde{u}_n(r) - g(\xi)\right)d\xi, n \geq 0 \qquad (3.149)$$

where γ is a general Lagrange multiplier that can be determined optimally via the variational theory as shown before. Recall that the (VIM) is used for ODEs and integro-differential equations and this can be obtained by differentiating the Volterra integral equation.

Moreover, the zeroth component $u_0(x)$ in Eq.(3.149) can be selected according to the order of the resulted ODE. For example, for ODEs of first order, second order, and third order, we select

$$\left.\begin{aligned} u_0(x) &= u(0) \\ u_0(x) &= u(0) + xu'(0) \\ u_0(x) &= u(0) + xu'(0) + \frac{1}{2!}x^2u''(0) \end{aligned}\right\} \quad (3.150)$$

respectively, and these are the first term, the first two terms, and the first three terms of the Taylor series of $u(x)$ at $x = 0$. The exact solution is thus given by Eq.(3.148).

In other words, to solve any Volterra integral equation by using this method, we should:

First: transform this equation to its equivalent ODE or its equivalent Volterra integro-differential equation, where Leibniz rule should be used.

The next step: consists of the determination of the Lagrange multiplier γ using the rules (i-v) given above.

Finally: we select the zeroth approximation $u_0(x)$ as indicated earlier. Having prepared all these steps, we then use the correction functional Eq.(3.149) to determine as many successive approximations as we can.

The variational iteration method will be illustrated by studying the following Volterra integral equations.

Example 1: Solve the Volterra integral equation by using the variational iteration method

$$u(x) = 1 - x + \int_0^x (x - t)u(t)\,dt, \quad (3.151)$$

Solution: Differentiating both sides of Eq.(3.151), and using Leibniz rule, we find

$$u'(x) = -1 + \int_0^x u(t)\,dt, \tag{3.152}$$

The initial condition $u(0) = 1$ is obtained by using $x = 0$ into Eq.(3.151), and hence we can select $u_0(x) = 1$.

The correction functional for equation Eq.(3.152) is

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) + 1 - \int_0^\xi u_n(r)\,dr \right) d\xi, \tag{3.153}$$

where we selected $\gamma = -1$ for the first order integro-differential equation Eq.(3.152). As stated before, we can use the initial condition to select $u_0(x) = 1$ that will lead to the following successive approximations

$$u_0(x) = 1, \tag{3.154}$$

$$u_1(x) = 1 - \int_0^x \left(u_0'(\xi) + 1 - \int_0^\xi u_0(r)\,dr \right) d\xi = 1 - x + \frac{1}{2!}x^2, \tag{3.155}$$

$$u_2(x) = 1 - x + \frac{1}{2!}x^2 - \int_0^x \left(u_1'(\xi) - \int_0^\xi u_1(r)\,dr \right) d\xi, \tag{3.156}$$

$$= 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4\,. \tag{3.157}$$

Recall that

$$u(x) = \lim_{n\to\infty} u_n(x) \tag{3.158}$$

$$= \lim_{n\to\infty} \left(1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \cdots + (-1)^n \frac{1}{n!}x^n \right) \tag{3.159}$$

that gives the exact solution by

$$u(x) = e^{-x} \,. \blacksquare \qquad (3.160)$$

Example 2: Solve the Volterra integral equation by using the variational iteration method

$$u(x) = 2\,sin(x) - \frac{1}{6}\,x^3 + \int_0^x (x-t)u(t)\,dt, \qquad (3.161)$$

Solution: Differentiating both sides of Eq.(3.161), once with respect to *x* and using Leibniz rule, we obtain the integro-differential equation

$$u'(x) = 2\,cos(x) - \frac{1}{2}\,x^2 + \int_0^x u(t)\,dt, \qquad (3.162)$$

where by using $x = 0$ into Eq.(3.161) we find $u(0) = 0$.
The (VIM) admits the use of correction functional for equation (3.162) by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - 2\,cos(x) + \frac{1}{2}\xi^2 - \int_0^\xi u_n(r)\,dr \right) d\xi \,. \qquad (3.163)$$

Proceeding as before, where we selected $\gamma = -1$ for the first order integro-differential equation Eq.(3.162). As stated before, we can use the initial condition to select $u_0(x) = 0$ that will lead to the following successive approximations

$$u_0(x) = 0, \qquad (3.164)$$

$$u_1(x) = -\int_0^x \left(u_0'(\xi) - 2\,cos(x) + \frac{1}{2}\xi^2 - \int_0^\xi u_0(r)\,dr \right) d\xi = 2sin(x) - \frac{1}{6}x^3 \,, (3.165)$$

$$u_2(x) = 2sin(x) - \frac{1}{6}x^3 - \int_0^x \left(u_1'(\xi) - 2\,cos(x) + \frac{1}{2}\xi^2 - \int_0^\xi u_1(r)\,dr \right) d\xi \quad (3.166)$$

$$= 2x - \frac{1}{6}x^3 - \frac{1}{120}x^5 \,. \tag{3.167}$$

$$u_3(x) = 2x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7. \tag{3.168}$$

and so on. The solution in a series form is given by

$$u(x) = x + \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right) \tag{3.169}$$

Recall that

$$u(x) = \lim_{n\to\infty} u_n(x), \tag{3.170}$$

that gives the exact solution by

$$u(x) = x + sin(x). \blacksquare \tag{3.171}$$

3.8 Comparison between Alternative Methods

Before making a comparison between all methods discussed in this chapter, it is convenient to point out that there are other techniques for solving Volterra integral equations that are beyond the scope of this text. However, when it comes to selecting a preferable method among the methods that were introduced in the previous sections, we cannot recommend a specific method.

Even though the method of reducing Volterra integral equation to initial value problem is rarely used, but it is the only method that may give directly the exact solution in a closed form. This is easily seen if the resulting initial value problem has constant coefficients. We may obtain the solution in a series form, similar to the results obtained by other methods, when the coefficients of the resulting initial value problem are functions of the independent variable *x*. The latter case will not be introduced here. A useful examples have been discussed in **Chapter 2**, **Section 5**.

However, we found that if the kernel $K(x,t)$ of the integral equation is a degenerate one that consists of a polynomial of one or two terms, the series method and the decomposition

method might be the best choices because it minimize the volume of calculations. The series solution obtained by using these methods might yield the exact solution in a closed form or we may obtain an approximation of the solution. Moreover, If $f(x)$ is a transcendental function, the series solution method works easier than the decomposition method.

Comparing the Adomian decomposition method with the successive approximation method, it is evident the decomposition method is much easier in that we integrate always very few terms to obtain the successive components, whereas in the other method we integrate many terms to evaluate the successive approximations after selecting the zeroth approximation. Moreover, for linear Volterra integral equations, the variational iteration method works effectively as the Adomian method.

In closing this section, we point out that the method of successive substitutions suffers from the huge size of calculations, especially if the function $f(x)$ is a trigonometric or exponential function.

To achieve our goal of the comparison between these methods, we illustrate this comparison by solving the following Volterra integral equation by using all various methods.

Example 1: Solve the Volterra integral equation by using the variational iteration method

$$u(x) = 1 + \int_0^x u(t)\,dt, \tag{3.172}$$

by using the six alternative methods.

Solution: We will start with the Adomian decomposition method, then we use the variational method, the series solution method and so on for the other methods examined in this text.

(a) Adomian Decomposition Method:

As discussed earlier we set

$$u(x) = \sum_{n=0}^{\infty} u_n\,(x). \tag{3.173}$$

Substituting Eq.(3.173) into both sides of Eq.(3.172) we obtain

$$u_0(x) + u_1(x) + u_2(x) + \cdots = 1 + \int_0^x (u_0(t) + u_1(t) + u_2(t) + \cdots)\, dt\,. \quad (3.174)$$

As stated before, we have to set the zeroth component $u_0(x)$ by all terms outside the integral sign, hence we have

$$u_0(x) = 1, \quad (3.175)$$

The first component $u_1(x)$ may be obtained by

$$u_1(x) = \int_0^x u_0(t)\, dt, \quad (3.176)$$

which gives the first component

$$u_1(x) = x, \quad (3.177)$$

Proceeding in the same manner we can easily obtain

$$u_2(x) = \frac{1}{2!}x^2, \quad (3.178)$$

and so on for other components. Noting that in the decomposition method we have

$$u(x) = u_0 + u_1 + u_2 + u_3 + \cdots, \quad (3.179)$$

hence by using the results Eqs.(177)-(178) into Eq.(179) we obtain the solution in a series form

$$u(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots, \quad (3.180)$$

and in a closed form by the exact solution

$$u(x) = e^x. \ \blacksquare \quad (3.181)$$

(b) The Variational Iteration Method:

Differentiating both sides of the integral equation Eq.(3.172) with respect to *x* gives

$$u'(x) = u(x). \tag{3.182}$$

Using the correction functional

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - u_n(\xi)\right) d\xi, \qquad n \geq 0, \tag{3.183}$$

gives the following approximations

$$u_0(x) = 1, \tag{3.184}$$

$$u_1(x) = 1 - \int_0^x \left(u_0'(t) - u_0(t)\right) dt = 1 + x, \tag{3.185}$$

$$u_2(x) = 1 + x - \int_0^x \left(u_1'(t) - u_1(t)\right) dt = 1 + x + \frac{1}{2!}x^2, \tag{3.186}$$

$$u_3(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots, \tag{3.187}$$

and so on. The solution in a series form is thus given by

$$u(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots, \tag{3.188}$$

that converges to the exact solution

$$u(x) = e^x. \blacksquare \tag{3.189}$$

(c) The Series Method:

As indicated before, assuming that $u(x)$ is **analytic**, hence we may write

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \ . \tag{3.190}$$

Substituting Eq.(3.190) into both sides of Eq.(3.172) we find

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 + a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3 + \cdots \tag{3.191}$$

Equating the coefficients of like powers of x from both sides we find

$$a_0 = 1, \tag{3.192}$$

$$a_1 = 1, \tag{3.193}$$

$$a_2 = \frac{1}{2!}, \tag{3.194}$$

$$a_3 = \frac{1}{3!}, \tag{3.195}$$

and so on. Substituting the results obtained for $a_k, k \geq 0$ into Eq.(190) we obtain the solution in a series form

$$u(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \ . \tag{3.196}$$

and in a closed form

$$u(x) = e^x. \tag{3.197}$$

is the exact solution of the example under discussion. ■

(d) Converting to Initial Value Problems:

Differentiating both sides of Eq.(172) with respect to x we obtain

$$u'(x) = u(x), \tag{3.198}$$

with the initial condition

$$u(0) = 1. \tag{3.199}$$

Solving the first separable order differential equation Eq.(3.198) and using the initial condition Eq.(3.199) the exact solution

$$u(x) = e^x. \tag{3.200}$$

is readily obtained.

(e) Successive Approximations Method:

In this method we select the zeroth approximation by

$$u_0(x) = 1. \tag{3.201}$$

Following the technique that was discussed above, the other approximations of the solution $u(x)$ can be easily obtained by

$$u_1(x) = 1 + x, \tag{3.202}$$

$$u_2(x) = 1 + x + \frac{1}{2!}x^2, \tag{3.203}$$

$$u_3(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3, \tag{3.204}$$

and so on. Accordingly, the *nth* component is given by

$$u_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n. \tag{3.205}$$

Consequently we find

$$u(x) = \lim_{n\to\infty} u_n(x), \tag{3.206}$$

$$= \lim_{n\to\infty} \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right) \tag{3.207}$$

$$= e^x, \tag{3.208}$$

the same solution obtained above. ■

(f) The Method of Successive Substitutions:

In this method we have to set $K(x,t)=1,\ \lambda=1,$ and $f(x)=1,$ hence we have

$$u(x)=1+1\int_0^x 1\,dt+\int_0^x\int_0^t 1\,dt_1dt+\int_0^x\int_0^t\int_0^{t_1} 1\,dt_2dt_1dt+\cdots \qquad (3.209)$$

$$=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\cdots \qquad (3.210)$$

$$=e^x, \qquad (3.211)$$

the same result obtained by other methods. ■

An important observation, and not a recommendation, can be made from the comparison performed above which suggests that the decomposition method, the variational iteration method, and the series method introduce promising improvements over other existing techniques.

We point out that there are other powerful techniques that are used for solving Volterra integral equations. Examples of these techniques include wavelet method, Legendre wavelets method, and many others.

3.9 Volterra Integral Equations of the First Kind

In this section we will study the Volterra integral equation of the first kind with separable kernel given by

$$f(x)=\int_0^x K(x,t)\,u(t)dt. \qquad (3.212)$$

We point out that the Adomian decomposition method or the variational iteration method **cannot be used in a straightforward manner**. However, the traditional series solution method that we used in this chapter can be used in a direct manner. Although there are other analytical and numerical methods that can be used to handle Volterra integral equations of the first kind, but in this section we will concern ourselves with two practical methods.

► 3.9.1 The Series Solution Method

The method was presented earlier. In this method we use the Taylor series for the solution $u(x)$ in the form

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \ . \tag{3.213}$$

where $a_n, n \geq 0$ are coefficients that will be determined. We then substitute this series into both sides of the Volterra integral equations of the first kind. We next evaluate the integral at the right side. By equating the coefficients of like terms into both sides we determine the coefficients $a_n, n \geq 0$, and this in turn leads to the solution in a series form. This will be illustrated by studying the following two examples.

Example 1: Solve the Volterra integral equation of the first kind by using the series solution method

$$x^2 + \frac{1}{6}x^3 = \int_0^x (2 + x - t)\, u(t) dt. \tag{3.214}$$

Solution: Substituting the series (3.213) in the right side and evaluating the resulting integral we find

$$x^2 + \frac{1}{6}x^3 = 2a_0 x + \left(\frac{1}{2}a_0 + a_1\right)x^2 + \left(\frac{1}{3!}a_1 + \frac{2}{3}a_2\right)x^3 + \cdots \tag{3.215}$$

Equating the coefficients of like terms from both sides gives

$$\left.\begin{array}{c} a_0 = 0 \\ a_1 = 1 \\ a_2 = a_3 = a_4 = \cdots = 0 \end{array}\right\} \tag{3.216}$$

Substituting this result into Eq.(3.213) gives the exact solution

$$u(x) = x. \blacksquare \tag{3.217}$$

Example 2: Find the solution of the Volterra equation of the first kind by using the series solution method

$$xe^x = \int_0^x e^{x-t}\, u(t)dt, \tag{3.218}$$

or equivalently

$$x = \int_0^x e^{-t}\, u(t)dt. \tag{3.219}$$

Solution: Substituting the series (3.213) in the right side and evaluating the resulting integral we find

$$x = a_0 x + \left(-\frac{1}{2}a_0 + \frac{1}{2}a_1\right)x^2 + \left(\frac{1}{6}a_0 - \frac{1}{3}a_1 + \frac{1}{3}a_2\right)x^3 + \cdots \tag{3.220}$$

Equating the coefficients of like terms from both sides gives

$$\left.\begin{array}{c} a_0 = 1 \\ a_1 = 1 \\ a_2 = \dfrac{1}{2!}, \quad a_3 = \dfrac{1}{3!}, \quad a_4 = \dfrac{1}{4!}, \ldots \end{array}\right\} \tag{3.221}$$

Substituting this result into Eq.(3.213) gives the exact solution

$$u(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots, \tag{3.222}$$

that converges to the exact solution

$$u(x) = e^x. \blacksquare \tag{3.223}$$

To apply the Adomian method or the variational iteration method, we should convert the Volterra integral equation of the **first kind** to a Volterra integral equation of the **second kind**.

► 3.9.2 Conversion of First Kind to Second Kind

It is important to note that the Volterra integral equation of the first kind can be handled simply by converting this equation to a Volterra equation of the second kind. This goal can be accomplished by differentiating both sides of Eq.(3.212) with respect to *x* to obtain

$$f'(x) = K(x,x)u(x) + \int_0^x K_x(x,t)u(t)\,dt, \tag{3.224}$$

by using Leibniz rule. If $K(x,x) \neq 0$ in the interval of discussion, then dividing both sides of Eq.(3.224) by *K*(*x*, *x*) yields

$$u(x) = \frac{f'(x)}{K(x,x)} - \frac{1}{K(x,x)} \int_0^x K_x(x,t)u(t)\,dt, \tag{3.225}$$

a Volterra integral equation of the second kind.

The case where the kernel $K(x,x) = 0$ leads to a complicated behavior of the problem that will not be investigated here.
To solve Eq.(3.225) we select any method that we discussed before. The technique of differentiating both sides of Volterra integral equation of the first kind, verifying that $K(x,x) \neq 0$, reducing to Volterra integral equation of the second kind and solving the resulting equation will be illustrated by discussing the same two examples solved by using the series solution method.

Example 3: Find the solution of the Volterra integral equation of the first kind

$$x^2 + \frac{1}{6}x^3 = \int_0^x (2 + x - t)\, u(t)dt. \tag{3.226}$$

Solution: We note first that $K(x,t) = 2 + x - t$, hence $K(x,x) \neq 0$. Differentiating both sides of Eq.(3.226) with respect to x yields

$$2x + \frac{1}{2}x^2 = 2u(x) + \int_0^x u(t)\, dt, \tag{3.227}$$

or equivalently

$$u(x) = x + \frac{1}{4}x^2 - \frac{1}{2}\int_0^x u(t)\, dt. \tag{3.228}$$

We prefer to use the *modified decomposition method*, hence we set

$$u_0(x) = x, \tag{3.229}$$

which gives

$$u_1(x) = \frac{1}{4}x^2 - \frac{1}{2}\int_0^x t\, dt = 0. \tag{3.230}$$

Accordingly, other components $u_n(x) = 0, n \geq 2$. The exact solution $u(x)$ is given by

$$u(x) = x, \tag{3.231}$$

obtained upon using the components obtained above. ■

Example 4: Find the solution of the Volterra equation of the first kind

$$xe^x = \int_0^x e^{x-t}\, u(t)dt, \tag{3.232}$$

or equivalently

$$x = \int_0^x e^{-t}\, u(t)dt. \tag{3.233}$$

Solution: We note first that $K(x,t) = e^{x-t}$, hence $K(x,x) = 1 \neq 0$. Differentiating both sides of Eq.(3.232) with respect to x yields

$$xe^x + e^x = u(x) + \int_0^x e^{x-t}\, u(t)dt, \tag{3.234}$$

or equivalently

$$u(x) = xe^x + e^x - \int_0^x e^{x-t}\, u(t)dt. \tag{3.235}$$

We shall solve the resulting equation by the *Adomian decomposition method* and by the *modified decomposition method*. We first start by applying the Adomian decomposition method, therefore we set

$$u_0(x) = e^x + xe^x. \tag{3.236}$$

Consequently, we obtain

$$u_1(x) = -\int_0^x e^{x-t}\,(e^t + te^t)dt = -e^x\left(x + \frac{1}{2!}x^2\right), \tag{3.237}$$

and

$$u_2(x) = \int_0^x e^{x-t}\left(te^t + \frac{t^2}{2!}e^t\right)dt = e^x\left(\frac{x^2}{2!} + \frac{x^3}{3!}\right), \tag{3.238}$$

and so on. Using the above results of the components obtained gives

$$u(x) = e^x\left(1 + x - x - \frac{x^2}{2!} + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^3}{3!} - \cdots\right) = e^x, \tag{3.239}$$

the exact solution obtained upon cancelling like terms with opposite signs. ■

Using the *modified decomposition method* we set

$$u_0(x) = e^x, \tag{3.240}$$

which gives

$$u_1(x) = xe^x - \int_0^x e^x\,dt = 0. \tag{3.241}$$

It immediately follows that

$$u_n(x) = 0, n \geq 1. \tag{3.242}$$

Accordingly, the exact solution is

$$u(x) = e^x. \blacksquare \tag{3.243}$$

3.10 Remarks Regarding Volterra Integral Equations of the First Kind

The relation between Volterra integral equations of the first and the second kind can be established in the following manner. The first kind Volterra equation is usually written as

$$f(x) = \int_0^x K(x,t)\,u(t)dt. \tag{3.244}$$

If the derivatives $\frac{df}{dx} = f'(x)$, $\frac{\partial K}{\partial x} = K_x(x,t)$, and $\frac{\partial K}{\partial t} = K_t(x,t)$ exist and are continuous, the equation can be reduced to one of the second kind in two ways.

The first and the simple way is to differentiate both sides of Eq. (3.244) with respect to x and we obtain the equation by using the Leibnitz rule

$$f'(x) = K(x,x)u(x) + \int_0^x K_x(x,t)\,u(t)dt. \qquad (3.245)$$

If $K(x,x) \neq 0$, then dividing throughout by this we obtain

$$\frac{f'(x)}{K(x,x)} = u(x) + \int_0^x \frac{K_x(x,t)}{K(x,x)}u(t)dt. \qquad (3.246)$$

and the reduction is accomplished. Thus, we can use the method already given above.

The second way to obtain the second kind Volterra integral equation from the first kind is by using integration by parts, if we set

$$\int_0^x u(t)\,dt = \phi(x), \qquad (3.247)$$

or equivalently

$$\int_0^t u(\xi)\,d\xi = \phi(t), \qquad (3.248)$$

We obtain the equation by integration by parts,

$$\left[K(x,t)\int_0^t u(\xi)\,d\xi\right]_{t=0}^{x} - \int_0^x K_t(x,t)\left(\int_0^t u(\xi)\,d\xi\right)dt = f(x) \qquad (3.249)$$

which reduces to

$$[K(x,t)\phi(t)]_{t=0}^{x} - \int_0^x K_t(x,t)\,\phi(t)dt = f(x) \tag{3.250}$$

$$K(x,x)\phi(x) - K(x,0)\phi(0) - \int_0^x K_t(x,t)\,\phi(t)dt = f(x) \tag{3.251}$$

It is obvious that $\phi(0) = 0$, and dividing out by $K(x,x)$ we have

$$\phi(x) = \left\{\frac{f(x)}{K(x,x)}\right\} + \int_0^x \frac{K_t(x,t)}{K(x,x)}\phi(t)dt \tag{3.252}$$

$$= F(x) + \int_0^x G(x,t)\,\phi(t)dt \tag{3.253}$$

where

$$F(x) = \frac{f(x)}{K(x,x)}, \qquad G(x,t) = \frac{K_t(x,t)}{K(x,x)} \tag{3.254}$$

For this second process it is apparently **not necessary** for $f(x)$ to be **differentiable**. However, the function $u(x)$ must finally be calculated by differentiating the function $\phi(x)$ given by the formula

$$\phi(x) = \left\{\frac{f(x)}{K(x,x)}\right\} + \int_0^x H(x,t;1)\left\{\frac{f(t)}{K(t,t)}\right\}dt \tag{3.255}$$

where $H(x,t;1)$ is the **resolvent kernel** corresponding to $\frac{K_t(x,t)}{K(x,x)}$. To do this $f(x)$ must be differentiable.

► **Note 3.1**: If $K(x,x)$ vanishes at some point of the basic interval $a \le x \le b$, for instance at $x = 0$, then the equations (3.246) and (3.253) will have a peculiar character, essentially different from that of equation of the second kind. These equations are called by Picard **the equation of the third kind**. However, if $K(x,x)$ vanishes identically, it is sometimes possible to obtain an equation of the second kind by previous transformations.

► **A special case of Volterra integral equation**

In the second kind Volterra equation if the kernel $K(x,t)$ is assumed to be $K(x,t) = \frac{A(x)}{A(t)}$ such that the equation takes the form

$$u(x) = f(x) + \lambda \int_0^x \frac{A(x)}{A(t)} u(t)dt. \tag{3.256}$$

and upon dividing throughout by $A(x)$ yields

$$\left\{\frac{u(x)}{A(x)}\right\} = \left\{\frac{f(x)}{A(x)}\right\} + \lambda \int_0^x \frac{u(t)}{A(t)} dt. \tag{3.257}$$

Now define $\frac{u(x)}{A(x)} = u_1(x)$ and $\frac{f(x)}{A(x)} = f_1(x)$ and Eq. (3.257) can be written as

$$u_1(x) = f_1(x) + \lambda \int_0^x u_1(t)\, dt. \tag{3.258}$$

Assuming that $u_2(x) = \int_0^x u_1(t)\, dt$, Eq. (3.258) can be reduced to an ordinary differential equation

$$\frac{du_2}{dx} - \lambda\, u_2 = f_1(x), \tag{3.259}$$

the general solution of which can be obtained as

$$u_2(x) = \int_0^x e^{\lambda(x-t)}\, f_1(t)dt + C_1 \tag{3.260}$$

Using the initial condition $u_2(0) = 0$ at $x = 0$, the Eq. (3.260) reduces to

$$u_2(x) = \int_0^x e^{\lambda(x-t)} f_1(t)dt. \qquad (3.261)$$

But $u_1(x) = \frac{du_2}{dx}$, and so the above equation can be reduced to an **integral equation** in terms of u_1 by differentiating according to the Leibnitz rule to yield

$$u_1(x) = \lambda \int_0^x e^{\lambda(x-t)} f_1(t)dt + f_1(x) \qquad (3.262)$$

Hence the **solution to the original problem** can be obtained multiplying throughout by $A(x)$,

$$u(x) = f(x) + \lambda \int_0^x e^{\lambda(x-t)} f(t)dt. \qquad (3.262)$$

Obviously, this formula can also be obtained by the previous method of **successive approximation**.

► **Note 3.2**: In **sec.(3.3)**, we have discussed the Volterra integral equation of the second kind of convolution type (**Faltung type**). If the kernel $K(x,t) \equiv K(x-t)$, then the **second kind and the first kind** will take the following forms, respectively,

$$u(x) = f(x) + \lambda \int_0^x K(x-t)\, f(t)dt. \qquad (3.263)$$

$$f(x) = \int_0^x K(x-t)\, f(t)dt. \qquad (3.264)$$

Equations (3.263) and (3.264) are very important special classes of Volterra integral equations, which Volterra called the equations of the closed cycle because the operator

$$V_x\{u(x)\} = \int_{-\infty}^{x} K(x,t)\, u(t)dt\,, \tag{3.265}$$

carries any periodic function $u(t)$ with arbitrary period T into another periodic function with the same period T, if and only if $K(x,t) = K(x-t)$. Today, they are usually called equations of the **Faltung type** because the operation

$$u * v = \int_{0}^{x} u(x-t)\, v(t)dt = \int_{0}^{x} u(t)\, v(x-t)dt, \tag{3.266}$$

is generally called **Faltung** (**convolution**) of the two functions u and v. In fact, setting $x - t = \tau$ such that $dt = -d\tau$, we obtain

$$u * v = \int_{0}^{x} u(\tau)\, v(x-\tau)d\tau = v * u\,. \tag{3.267}$$

Finally, we notice that it is often convenient to set $u * u = u*^2; u * u*^2 = u*^3$ and so on. Using the Faltung sign, the integral equations (3.263) and (3.264) can be written, respectively, as

$$u(x) = f(x) + \lambda\, K(x) * u(x)\,. \tag{3.268}$$

$$f(x) = K(x) * u(x)\,. \tag{3.269}$$

The main device in dealing with such equations is the **Laplace transformation**

$$\mathcal{L}\{u(x)\} = \int_{0}^{\infty} e^{-sx}\, u(x)dx\,, \tag{3.270}$$

because, under some restrictions, this operator transforms the convolution into an ordinary product

$$\mathcal{L}\{K(x) * u(x)\} = \mathcal{L}\{K(x)\}\mathcal{L}\{u(x)\}\,. \tag{3.271}$$

In this way, **the integral equation can be transformed into an algebraic equation**, and then solving $\mathcal{L}\{u(x)\}$, and inverting the transform we obtain the desired solution.

Example 1: Solve the convolution type Volterra integral equation

$$u(x) = x \int_0^x (t - x)\, u(t) dt\,. \tag{3.272}$$

Solution: By taking the Laplace transform of the given equation, we have

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{x\} - \mathcal{L}\{x\}\mathcal{L}\{u(x)\}\,, \tag{3.273}$$

which reduces to

$$\mathcal{L}\{u(x)\} = \frac{1}{s^2 + 1}\,. \tag{3.274}$$

The inverse is given by

$$u(x) = sin(x)\,, \tag{3.275}$$

which is the required solution. ■

Example 2: Solve the Abel's problem

$$f(x) = \int_0^x \frac{1}{\sqrt{x - t}} u(t) dt\,. \tag{3.276}$$

Solution: Taking the Laplace transform of the given equation yields

$$\mathcal{L}\{f(x)\} = \mathcal{L}\left\{\frac{1}{\sqrt{x}} \mathcal{L}\{u(x)\}\right\}, \tag{3.277}$$

This transformed equation reduces to

$$\mathcal{L}\{u(x)\} = \sqrt{\frac{s}{\pi}}\,\mathcal{L}\{f(x)\}. \tag{3.278}$$

Inversion of this problem is performed as follows:

$$u(x) = \frac{1}{\sqrt{\pi}}\mathcal{L}^{-1}\left\{\frac{s\mathcal{L}\{f(x)\}}{\sqrt{s}}\right\} \tag{3.279}$$

$$= \frac{1}{\sqrt{\pi}}\frac{d}{dx}\mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{f(x)\}}{\sqrt{s}}\right\} \tag{3.280}$$

$$= \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)dt}{\sqrt{x-t}} \tag{3.281}$$

which is the desired solution of the Abel equation. ■

► **Remark 3.3**: It is clear that the **Leibnitz rule is not applicable to the above solution** because the integrand is discontinuous at the interval of integration. To determine the solution in another form, we first integrate because the integrand is the known function.

Put $x - t = \tau^2,$ and with this substitution, it is obvious that

$$\int_0^x \frac{f(t)dt}{\sqrt{x-t}} = 2\int_0^{\sqrt{x}} f(x-\tau^2)\,d\tau\,. \tag{3.282}$$

Thus, we obtain

$$u(x) = \frac{2}{\pi}\frac{d}{dx}\int_0^{\sqrt{x}} f(x-\tau^2)\,d\tau\,. \tag{3.283}$$

Thus, using this information, the solution for $u(x)$ can be written as

$$u(x) = \frac{f(0)}{\sqrt{x}} + \int_0^x \frac{f'(t)}{\sqrt{x-t}}\, dt. \quad \blacksquare \tag{3.284}$$

► Volterra integral equation and linear differential equations

There is a fundamental relationship between Volterra integral equations and ordinary differential equations. In fact, the solution of any differential equation of the type

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n(x)y = F(x), \tag{3.285}$$

with continuous coefficients, together with the initial conditions

$$y(0) = c_0, y'(0) = c_1, y''(0) = c_2, \ldots, y^{n-1}(0) = c_{n-1}\,. \tag{3.286}$$

can be reduced to the solution of a certain Volterra integral equation of the second kind

$$f(x) = u(x) + \int_0^x K(x,t)\, u(t)dt. \tag{3.287}$$

In order to achieve this, let us make the transformation

$$\frac{d^n y}{dx^n} = u(x)\,. \tag{3.288}$$

Hence integrating with respect to x from 0 to x

$$\frac{d^{n-1}y}{dx^{n-1}} = \int_0^x u(t)\, dt + c_{n-1}\,. \tag{3.289}$$

Thus, the successive integrals are

$$\frac{d^{n-2}y}{dx^{n-2}} = \int_0^x \int_0^x u(t)\, dt\, dt + c_{n-1}x + c_{n-2}\,, \tag{3.290}$$

$$\frac{d^{n-3}y}{dx^{n-3}} = \int_0^x \int_0^x \int_0^x u(t)\,dt\,dt\,dt + c_{n-1}x^2 + c_{n-2}x + c_{n-3}\,, \qquad (3.291)$$

… … … … = … … … …

Proceeding in this manner we obtain

$$y(x) = \underbrace{\int_0^x \int_0^x \int_0^x \dots \int_0^x}_{n\ fold\ integration} \int_0^x u(t)\,dtdtdt \dots dt$$

$$+c_{n-1}\frac{x^{n-1}}{(n-1)!} + c_{n-2}\frac{x^{n-2}}{(n-2)!} + \cdots + c_1 x + c_0 \quad (3.292)$$

$$= \int_0^x \frac{(x-t)^{n-1}}{(n-1)!}u(t)dt + c_{n-1}\frac{x^{n-1}}{(n-1)!} + c_{n-2}\frac{x^{n-2}}{(n-2)!} + \cdots + c_1 x + c_0 \qquad (3.293)$$

Returning to the differential equation (3.285), we see that it can be written as

$$u(x) + \int_0^x \left\{a_1(x) + a_2 x(x-t) + a_3(x)\frac{(x-t)^2}{2!} + \cdots + a_n(x)\frac{(x-t)^{n-1}}{(n-1)!}\right\}u(t)dt \quad (3.294)$$

$$= F(x) - c_{n-1}a_1(x) - (c_{n-1} + c_{n-2})a_2(x) - \cdots - \left(c_{n-1}\frac{x^{n-1}}{(n-1)!} + \cdots + c_1 x + c_0\right)a_n(x)$$

and this reduces to Eq. (3.287)

where

$$K(x,t) = \sum_{v=1}^{n} a_v\,(x)\frac{(x-t)^{v-1}}{(v-1)!}\,, \qquad (3.295)$$

and

$$f(x) = F(x) - c_{n-1}a_1(x) - (c_{n-1} + c_{n-2})a_2(x)$$

$$-\left(c_{n-1}\frac{x^{n-1}}{(n-1)!} + \cdots + c_1 x + c_0\right)a_n(x). \qquad (3.296)$$

Conversely, the solution, i.e. Eq. (3.287) with K and f given by Eqs. (3.295) and (3.296) and substituting values for $u(x)$ in the last equation of equation (3.293), we obtain the (**unique**) solution of Eq. (3.285) which satisfies the initial conditions (Eq. (3.286)). If the leading coefficient in equation (3.285) is not unity but $a_0(x)$ Eq. (3.287) becomes

$$a_0(x)u(x) + \int_0^x K(x,t)\,u(t)dt = f(x)\,. \qquad (3.297)$$

where K and f are still given by Eqs. (3.295) and (3.296), respectively.

► **Remark 3.4**: If $a_0(x) \neq 0$, in the interval considered, nothing is changed; however, if $a_0(x)$ vanishes at some point, we see that an equation of the type (3.297) is equivalent to a singular differential equation, at least, when $K(x,t)$ is a polynomial in t.

Exercise: See chapter 2, sec.6.

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Chapter 4

Fredholm Integral Equations

4.1 Introduction

In this chapter, we shall be concerned with the **Fredholm integral equations (FIE)**. In the previous chapters we have clearly defined the integral equations with some useful illustrations. In addition, we have discussed the evolution of integral equations that have enormous applications in the physical problems. Many initial and boundary value problems can be transformed into integral equations. It was also shown that FIE can be derived from boundary value problems. Problems in the mathematical physics are usually governed by the integral equations. There are quite a few categories of the integral equations. Singular integral equations are very useful in many physical problems such as elasticity, fluid mechanics, electromagnetic theory, and so on.

In 1825 Abel first encountered a singular integral equation in connection with the famous tautochrone problem, namely the determination of the shape of the curve with given end points along which a particle slides under the action of gravity in a given interval of time. An equation of an unknown function $u(x)$, of a single variable x in the domain $a \leq x \leq b$ is said to be an integral equation (IE) for $u(x)$, if $u(x)$ appears under the integral sign and the integral exists in some sense. As stated before, in FIE, the integral containing the unknown function $u(x)$ is characterized by fixed (constant) limits of the integration. For examples,

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)\,dt, \qquad a \leq x \leq b \tag{4.1}$$

$$f(x) = \int_a^b K(x,t)u(t)\,dt, \qquad a \leq x \leq b \tag{4.2}$$

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\{u(t)\}^2\, dt, \qquad a \le x \le b \qquad (4.3)$$

are all **Fredholm integral equations (FIEs)**. Eq. (4.1) is defined as the nonhomogeneous Fredholm linear integral equation of the second kind; Eq. (4.2) is the Fredholm linear integral equation of the first kind; and Eq. (4.3) is the Fredholm nonlinear integral equation of the second kind.

In all these examples, $K(x,t)$ and $f(x)$ are known functions. $K(x,t)$ is called the kernel of the integral equation defined in the rectangle R, for which $a \le x \le b$ and $a \le t \le b$ and $f(x)$ is called the forcing term defined in $a \le x \le b$. If $f(x) = 0$, then the equations are **homogeneous**. The functions u, f, K may be complex-valued functions. The linear and nonlinear integral equations are defined when the unknown function appears linearly or nonlinearly under the integral sign. The parameter λ is a known quantity.

A considerable amount of discussion will be directed towards the various methods and techniques that are used for solving Eq. (4.1) starting with the most recent methods that proved to be highly reliable and accurate. We will mostly use ***degenerate*** or ***separable*** kernels. The standard form of the *degenerate* or *separable* kernel is given by

$$K(x,t) = \sum_{i=1}^{n} f_i(x)\, g_i(t)\,. \qquad (4.4)$$

The expressions $x - t\,, x + t\,, (x-t)^2\,, 3xt\,, x^2 - 3xt + t^2\,, etc.$ are examples of separable kernels. For other well-behaved **non-separable** kernels, we can convert it to separable in the form (4.4) simply by expanding these kernels using Taylor's expansion.

4.2 Various types of Fredholm integral equations

Associated with the integral Eq. (4.1), the following meaning of various forms are used

♦ If the kernel $K(x,t)$ is such that

$$\int_a^b \int_a^b |K(x,t)|^2\, dtxdt < \infty\,, \qquad (4.5)$$

has a finite value; i.e. the kernel $K(x,t)$ is **square integrable** in the rectangle R; then we call the kernel a **regular kernel** and the corresponding integral is called a **regular integral equation**. This condition gives rise to the development of the solution of the FIE (4.1).

♦ If the kernel $K(x,t)$ is of the form

$$K(x,t) = \frac{H(x,t)}{|x-t|^{\alpha}}, \tag{4.6}$$

where $H(x,t)$ is bounded in R, $a \le x \le b$ and $a \le t \le b$ with $H(x,t) \neq 0$, and α is a constant such that $0 \le \alpha \le 1$, then the integral equation is said to be a **weakly singular** integral equation.

♦ If the kernel $K(x,t)$ is of the form

$$K(x,t) = \frac{H(x,t)}{x-t}, \tag{4.7}$$

where $H(x,t)$ is a differentiable function of (x,t) with $H(x,t) \neq 0$, then the integral equation is said to be a **singular equation with Cauchy kernel** where the integral $\int_a^b \frac{H(x,t)}{x-t} u dt$ is understood in the sense of **Cauchy Principal Value** (CPV) and the notation P.V. b a is usually used to denote this. Thus

$$P.V. \int_a^b \frac{u(t)}{x-t} dt = \lim_{\varepsilon \to 0} \left\{ \int_a^{x-\varepsilon} \frac{u(t)}{x-t} dt + \int_{x+\varepsilon}^b \frac{u(t)}{x-t} dt \right\}. \tag{4.8}$$

♦ If the kernel $K(x,t)$ is of the form

$$K(x,t) = \frac{H(x,t)}{(x-t)^2}, \tag{4.9}$$

where $H(x,t)$ is a differentiable function of (x,t) with $H(x,t) \neq 0$, then the integral equation is said to be a **hyper-singular** integral equation when the integral

$$\int_a^b \frac{H(x,t)}{(x-t)^2} u(t) dt, \tag{4.10}$$

is to be understood as a *hyper-singular integral* and the notation $\oint_a^b$ *is* usually used to denote this. Thus,

$$\oint_a^b \frac{u(t)}{(x-t)^2}\,dt \approx \lim_{\varepsilon\to 0}\left\{\int_a^{x-\varepsilon} \frac{u(t)}{(x-t)^2}\,dt + \int_{x+\varepsilon}^b \frac{u(t)}{(x-t)^2}\,dt - \frac{u(x+\varepsilon)+u(x-\varepsilon)}{2\varepsilon}\right\}. \quad (4.11)$$

♦ If the limits of the integral, a and b are constants, then it is a **Fredholm integral equation**. Otherwise, if a or b is a variable x, then the integral equation is called **Volterra integral equation**.

In the following section we shall discuss the various methods of solutions of the FIEs. However, before we start discussing these methods, it is also convenient to state, without proof, the so-called *Fredholm Alternative Theorem* that relates the solutions of homogeneous and nonhomogeneous FIEs.

Theorem 4.1: (Fredholm Alternative Theorem)

The nonhomogeneous FIE (4.1) has one and only one solution if the only solution to the homogeneous FIE

$$u(x) = \lambda \int_a^b K(x,t)u(t)\,dt, \qquad a \le x \le b \quad (4.12)$$

is the trivial solution $u(x) = 0$.

We end this section by introducing the **necessary condition** that will guarantee a unique solution to the integral equation (4.1) in the interval of discussion.

Theorem 4.2: (Unique Solution)

Considering Eq. (4.4); if the kernel $K(x,t)$ is real, continuous and bounded in the square $a \le x \le b$ and $a \le t \le b$, i.e. if

$$|K(x,t)| \le M, \qquad a \le x \le b\,, a \le t \le b \quad (4.13)$$

and if $f(x) \neq 0$, and continuous in $a \leq x \leq b$, then the necessary condition that will guarantee that Eq.(4.1) has only a unique solution is given by

$$|\lambda| M(b-a) < 1. \tag{4.14}$$

Remark 4.3: It is important to note that a continuous solution to FIE may exist, even though the condition (4.14) is not satisfied. That is, **if the necessary condition (4.14) does not hold**, then **a continuous solution may exist** for FIE. To illustrate this, we give the following two examples.

Example 1: Consider the FIE

$$u(x) = -4 + \int_0^1 (2x + 3t)u(t)\, dt\,. \tag{4.15}$$

In this example, $\lambda = 1, |K(x,t)| \leq 5$, and $(b-a) = 1$; therefore

$$|\lambda| M(b-a) = 5 \not< 1. \tag{4.16}$$

Accordingly, the necessary condition (4.14) fails to hold, but in fact the integral equation (4.15) has an exact solution given by

$$u(x) = 4x\,. \tag{4.17}$$

and this can be justified through direct substitution.

Example 2: Consider the FIE

$$u(x) = 2 - 3x + \int_0^1 (3x + t)u(t)\, dt\,. \tag{4.18}$$

In this example, $\lambda = 1, |K(x,t)| \leq 4$, and $(b-a) = 1$; therefore

$$|\lambda| M(b-a) = 4 \not< 1. \tag{4.19}$$

However, the FIE (4.18) has an exact solution given by

$$u(x) = 6x\,. \tag{4.20}$$

As indicated, we will pay more attention to the practical techniques for solving integral equations rather than the abstract theorems. A variety of **analytic** and **numerical** methods have been used to handle FIEs. The *direct computation method*, the *successive approximations method*, and *converting FIE to an equivalent BVP* are among many traditional methods that were commonly used. However, in the following we will start discussing the most recently developed methods, namely, the *Adomian decomposition method* (ADM), *the modified Adomian decomposition method* (MADM), and the *variation iteration method* (VIM) that handle successfully the FIEs of the second kind.

4.3 The Adomian Decomposition Method (ADM)

As stated earlier, Adomian developed the *Adomian decomposition method* or simply the *decomposition method* that proved to work for all types of differential, integral and integro differential equations, linear or nonlinear. The method was introduced by Adomian in his books **[2]** and **[3]** and other related research papers such as **[4]** and **[5]** that were mentioned in Chapter 3. The focus of the two books was mainly on ordinary and partial differential equations. The method has much similarity with the ***Neumann series*** as will be discussed in a forthcoming sections. Like Neumann series method, the method provides the solution in a series form. In fact, we have seen from **Chapter 3** that the decomposition method mostly establishes the solution in the form of a power series. The approach we will follow here is identical to the same approach that was implemented earlier in **Chapter 3**.

In this method, the solution $u(x)$ will be decomposed into an infinite series of components, that will be determined, given by the series form

$$u(x) = \sum_{n=0}^{\infty} u_n(x)\,, \tag{4.21}$$

with u_0 identified by all terms out of the integral sign, i.e.

$$u_0(x) = f(x)\,. \tag{4.22}$$

The main idea here like perturbation technique is to determine the zeroth decomposition $u_0(x)$ by the known function $f(x)$. Once $u_0(x)$ is known, then successively determine $u_1(x), u_2(x), u_3(x),$ and so on.

Now, substituting Eq.(4.21) in both sides of Eq.(4.1) yields

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^b K(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt \qquad (4.23)$$

or equivalently

$$\begin{aligned} u_0(x) + u_1(x) + u_2(x) + \cdots = {} & f(x) + \lambda \int_a^b K(x,t)\, u_0(t)dt \\ & + \lambda \int_a^b K(x,t)\, u_1(t)dt \\ & + \lambda \int_a^b K(x,t)\, u_2(t)dt \\ & + \cdots \end{aligned} \qquad 4.24$$

The components $u_0(x)$, $u_1(x)$, $u_2(x)$, $u_3(x)$, ... of the unknown function $u(x)$ are completely determined in a recurrent manner if we set

$$u_0(x) = f(x)\,, \qquad (4.25)$$

$$u_1(x) = \lambda \int_a^b K(x,t)\, u_0(t)dt\,, \qquad (4.26)$$

$$u_2(x) = \lambda \int_a^b K(x,t)\, u_1(t)dt\,, \qquad (4.27)$$

$$u_3(x) = \lambda \int_a^b K(x,t)\, u_2(t)dt\,, \qquad (4.28)$$

and so on. The above discussed scheme for the determination of the components $u_0(x)$, $u_1(x)$, $u_2(x)$, $u_3(x)$, ... of the solution $u(x)$ of Eq. (4.1) can be written in a recursive manner by

$$u_0(x) = f(x)\,, \tag{4.29}$$

$$u_{n+1}(x) = \lambda \int_a^b K(x,t)\,u_n(t)dt\,, n \geq 1 \tag{4.30}$$

In view of Eqs. (4.29) and (4.30), the components $u_0(x)$, $u_1(x)$, $u_2(x)$, $u_3(x)$, ... follow immediately. With these components determined, the solution $u(x)$ of Eq. (4.1) is readily determined in a series form using the decomposition in Eq. (4.21).

It is important to note that, for some problems, the obtained series for $u(x)$ converges to the **exact** solution in a closed form if such a solution exists as will be seen later. However, for concrete problems, where exact solution cannot be evaluated, a truncated series $\sum_{n=0}^{k} u_n\,(x)$ is usually used to approximate the solution $u(x)$ and this can be used for **numerical** purposes. We point out here that few terms of the truncated series usually provide the higher accuracy level of the approximate solution if compared with the existing numerical techniques. This is because of the ***uniformly convergence property*** of the infinite series a few terms will attain the maximum accuracy.
The decomposition technique proved to be effective and reliable even if applied to nonlinear FIEs as will be discussed in a forthcoming chapter.

In the following we discuss some examples that illustrate the decomposition method outlined above.

Example 1: Solve the following FIE of the second kind by using the ADM

$$u(x) = \frac{9}{10}x^2 + \int_0^1 \frac{1}{2}x^2t^2dt\,, \tag{4.31}$$

Solution: It is clear that $f(x) = \frac{9}{10}x^2\,, \lambda = 1\,, K(x,t) = \frac{1}{2}x^2t^2$. To evaluate the components $u_0(x)$, $u_1(x)$, $u_2(x)$, $u_3(x)$, ... of the series solution, we use the recursive scheme (4.29) and (4.30) to find

$$u_0(x) = \frac{9}{10}x^2\,, \tag{4.32}$$

$$u_1(x) = \int_0^1 \frac{1}{2}x^2t^2\,u_0(t)dt = \int_0^1 \frac{1}{2}x^2\,\frac{9}{10}t^4\,dt = \frac{9}{100}x^2 \tag{4.33}$$

$$u_2(x) = \int_0^1 \frac{1}{2}x^2t^2\,u_1(t)dt = \int_0^1 \frac{1}{2}x^2\,\frac{9}{100}t^2\,dt = \frac{9}{1000}x^2 \tag{4.34}$$

and so on. Noting that

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots, \tag{4.35}$$

we can easily obtain the solution in a series form given by

$$u(x) = \frac{9}{10}x^2 + \frac{9}{100}x^2 + \frac{9}{1000}x^2 + \cdots, \tag{4.36}$$

so that the solution of Eq. (4.31) in a closed form

$$u(x) = x^2\,, \tag{4.37}$$

follows immediately upon using the formula for the sum of the infinite geometric series. ■

Example 2: Solve the following FIE of the second kind by using the ADM

$$u(x) = cos(x) + 2x + \int_0^\pi x\,tu(t)dt\,, \tag{4.38}$$

Solution: Proceeding as in **Example 1**, we set

$$u_0(x) = cos(x) + 2x\,, \tag{4.39}$$

$$u_1(x) = \int_0^\pi xt\,u_0(t)dt = \int_0^\pi xt(cos(t) + 2t)\,dt = \left(-2 + \frac{2}{3}\pi^3\right)x, \tag{4.40}$$

$$u_2(x) = \int_0^{\pi} xt\, u_1(t)dt = \int_0^{\pi} x\left(-2 + \frac{2}{3}\pi^3\right) t^2\, dt = \left(-\frac{2}{3}\pi^3 + \frac{2}{9}\pi^6\right)x, \qquad (4.41)$$

and so on. Noting that

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots, \qquad (4.42)$$

we can easily obtain the solution in a series form given by

$$u(x) = cos(x) + 2x\ + \left(-2 + \frac{2}{3}\pi^3\right)x + \left(-\frac{2}{3}\pi^3 + \frac{2}{9}\pi^6\right)x + \cdots, \qquad (4.43)$$

so that the solution of Eq. (4.38) in a closed form

$$u(x) = cos(x)\,, \qquad (4.44)$$

by eliminating the so-called self-cancelling noise terms between various components of $u(x)$. The answer obtained can be justified through substitution. The self-cancelling noise terms are defined to be similar terms with opposite signs that will vanish in the limit. The phenomenon of noise terms was mentioned in **Chapter 3**, Sec. 3.6.2. ■

It is important to note that the evaluation of the components $u_0(x), u_1(x), u_2(x)$, ... is simple as we observed from the examples above. However, we can still reduce the size of calculations by using a *modified version of the decomposition method*. In this modified approach, we often need to evaluate the first two components $u_0(x)$ and $u_1(x)$ only.

► 4.3.1 The Modified Decomposition Method

In the previous section we have seen that the (ADM) provides the solutions in an infinite series of components. The components $u_i\,, i \geq 0$ are easily computed if the *inhomogeneous* term $f(x)$ in the FIE

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)dt\,, \qquad (4.45)$$

consists of a polynomial of one or two terms. However, if the function $f(x)$ consists of a combination of two or more of polynomials, trigonometric function, hyperbolic functions, and other, the evaluation of the components $u_i\,, i \geq 0$ requires more work.

As stated earlier in Chapter 3, Sec. 3.6.1; sometimes it may be useful to decompose the $f(x)$ term into $f(x) = f_0(x) + f_1(x)$ such that

$$u_0(x) = f_0(x)\,, \tag{4.46}$$

$$u_1(x) = f_1(x) + \lambda \int_a^b K(x,t)\, u_0(t)dt\ , \tag{4.47}$$

$$u_2(x) = \lambda \int_a^b K(x,t)\, u_1(t)dt\ , \tag{4.48}$$

$$u_3(x) = \lambda \int_a^b K(x,t)\, u_2(t)dt\ , \tag{4.49}$$

$$\vdots$$

$$u_{n+1}(x) = \lambda \int_a^b K(x,t)\, u_n(t)dt\ , n \geq 1 \tag{4.50}$$

This decomposition is called ***modified decomposition method***.

It was shown that this modification facilitates the computational work and accelerates the convergence of the series solution. As presented before, the modified decomposition method (MDM) depends mainly on splitting the function $f(x)$ consists of only term.

It is obvious that the difference between the standard recurrence relation (4.30) and the modified recurrence relation (4.50) rests only in the formation of the first two components $u_0(x)$ and $u_1(x)$ only. The other components $u_i\,, i \geq 2$ remain the same in the two recurrence relations. Although this variation in the formation of $u_0(x)$ and $u_1(x)$ is slight, however it has been shown that it accelerates the convergence of the solution and minimizes the size of calculations. Moreover , reducing the number of terms in $f_0(x)$ affects not only the component $u_1(x)$, but also the other components as well.

Remark 4.5: We here emphasize on the following two important remarks.

► **First**, by proper selection of the functions $f_0(x)$ and $f_1(x)$, the exact solution $u(x)$ may be obtained by using very few iterations, and sometimes by evaluating only two components. The success of this can be made through trials only. A rule that may help for proper choice of $f_0(x)$ and $f_1(x)$ could not be found yet.
► **Second**, if $f(x)$ consists of one term only, the MDM **cannot** be used in this case.

We explain the (MDM) by the following illustrative examples:

Example 3: Solve the FIE by using the *modified decomposition method* (MDM)

$$u(x) = e^{3x} - \frac{1}{9}(2e^3 + 1)x + \int_0^1 x\,tu(t)dt\,, \tag{4.51}$$

Solution: To apply the modified decomposition scheme as discussed above, we first split the function $f(x)$ into

$$f_0(x) = e^{3x}\,, \tag{4.52}$$

and

$$f_1(x) = -\frac{1}{9}(2e^3 + 1)x\,. \tag{4.53}$$

Therefore, we set

$$u_0(x) = e^{3x}\,, \tag{4.54}$$

and

$$\begin{aligned} u_1(x) &= -\frac{1}{9}(2e^3 + 1)x + \int_0^1 xt\,u_0(t)dt \\ &= -\frac{1}{9}(2e^3 + 1)x + \int_0^1 t\,e^{3t}dt = 0\,. \end{aligned} \tag{4.55}$$

In view of Eq. (4.55), we conclude that $u_n(x) = 0, n \geq 1$. The exact solution

$$u(x) = e^{3x} . \tag{4.56}$$

follows immediately. ■

Example 4: Solve the FIE by using the *modified decomposition method* (MDM)

$$u(x) = sin^{-1}(x) + \left(\frac{\pi}{2} - 1\right)x - \int_0^1 x\, u(t)dt , \tag{4.57}$$

Solution: To apply the modified decomposition scheme as discussed above, we first split the function $f(x)$ into

$$f_0(x) = sin^{-1}(x) , \tag{4.58}$$

and

$$f_1(x) = \left(\frac{\pi}{2} - 1\right)x . \tag{4.59}$$

Therefore, we set

$$u_0(x) = sin^{-1}(x) , \tag{4.60}$$

and

$$u_1(x) = \left(\frac{\pi}{2} - 1\right)x + \int_0^1 x\, u_0(t)dt \tag{4.61}$$

$$= \left(\frac{\pi}{2} - 1\right)x + x\int_0^1 sin^{-1}(t)\, dt , \tag{4.62}$$

$$= 0 .$$

In view of Eq. (4.62), we conclude that $u_n(x) = 0, n \geq 1$. The exact solution

$$u(x) = sin^{-1}(x) . \tag{4.63}$$

is readily obtained. ■

Example 5: Solve the FIE by using the *modified decomposition method* (MDM)

$$u(x) = sin(x) + cos(x) - 2x + \frac{\pi}{2} + \int_0^{\frac{\pi}{2}} (x - t)\, u(t)dt\,, \qquad (4.64)$$

Solution: To apply the modified decomposition scheme as discussed above, we first split the function $f(x)$ into

$$f_0(x) = sin(x) + cos(x)\,, \qquad (4.65)$$

and

$$f_1(x) = -2x + \frac{\pi}{2}\,. \qquad (4.66)$$

Therefore, we set

$$u_0(x) = sin(x) + cos(x)\,, \qquad (4.67)$$

and

$$u_1(x) == -2x + \frac{\pi}{2} + \int_0^{\frac{\pi}{2}} (x - t)\, u_0(t)dt = 0. \qquad (4.68)$$

In view of Eq. (4.68), we conclude that $u_n(x) = 0, n \geq 1$. The exact solution

$$u(x) = sin(x) + cos(x)\,. \qquad (4.69)$$

is readily obtained. ■

This confirms our belief that the decomposition method and the modified decomposition method introduce the solution of FIEs in the form of a rapidly convergent power series with elegantly computable terms. However, if $f(x)$ consists of more than one term, the modified decomposition method minimizes the volume of the computational work.

► 4.3.2 The Noise Terms Phenomenon

As stated in **Chapter 3**, Se. 3.6.2; the noise terms phenomenon is a method which accelerates the convergence of the Adomian decomposition method. The noise terms are the identical terms with opposite signs that arise in the components, and may exist only for nonhomogeneous equations. The noise terms phenomenon demonstrates a useful tool for fast convergence of the solution. It was proved that the noise terms phenomenon may appear only for nonhomogeneous PDEs of any order or nonhomogeneous integral equations of any kind. The noise terms, if exist, it will appear in all components. It was found that by canceling the noise terms between $u_0(x)$ and $u_1(x)$, even though $u_1(x)$ contains other terms, the remaining non-canceled terms of $u_0(x)$ may give the exact solution of the equation. However, the appearance of the noise terms between $u_0(x)$ and $u_1(x)$ is not always sufficient to obtain the exact solution by canceling these noise terms. Therefore, it necessary to show that the non-canceled terms of $u_0(x)$ satisfy the given integral equation. In case the non-canceled terms of $u_0(x)$ did not satisfy the given equation, or the noise terms did not appear, then we proceed to determine more components of the series solution.

It was shown by many authors that non homogeneity condition is not sufficient to give noise terms. Moreover, it is necessary that the zeroth component $u_0(x)$ should contain the exact solution $u(x)$ among other terms, and this only may give noise terms about components. Moreover, noise terms may appear if the exact solution $u(x)$ is part of the zeroth component $u_0(x)$.

The phenomenon of the useful noise terms will be explained by the following illustrative examples.

Example 6: Solve the FIE by using noise terms phenomenon

$$u(x) = xcos(x) + 2x + \int_0^{\pi} x\,u(t)dt\,, \tag{4.70}$$

Solution: The Adomian method admits the use of the recurrence relation

$$u_0(x) = xcos(x) + 2x, \tag{4.71}$$

$$u_{k+1}(x) = \int_0^{\pi} xu_k(t)dt\,, \qquad k \geq 0. \tag{4.72}$$

This gives

$$u_0(x) = xcos(x) + 2x, \tag{4.73}$$

$$u_1(x) = \int_0^{\pi} xu_0(t)dt = -2x + \pi^2 x\,. \tag{4.74}$$

The noise terms $\mp 2x^2$ appear in $u_0(x)$ and $u_1(x)$. Canceling these terms from the zeroth component $u_0(x)$ gives the exact solution

$$u(x) = xcos(x), \tag{4.75}$$

that satisfies the integral equation. This should be justified by substitution. The other terms of $u_1(x)$ vanish in the limit with other terms of the other components. ■

Example 7: Solve the FIE by using noise terms phenomenon

$$u(x) = cos(x) - sin(x) + 2x - \frac{\pi}{2}x + \int_0^{\frac{\pi}{2}} x\,t\,u(t)dt\,, \tag{4.76}$$

Solution: The Adomian method admits the use of the recurrence relation

$$u_0(x) = cos(x) - sin(x) + 2x - \frac{\pi}{2}x, \tag{4.77}$$

$$u_{k+1}(x) = \int_0^{\frac{\pi}{2}} x\,t\,u_k(t)dt\,, \qquad k \geq 0. \tag{4.78}$$

This gives

$$u_0(x) = cos(x) - sin(x) + 2x - \frac{\pi}{2}x, \tag{4.79}$$

$$u_1(x) = \int_0^{\frac{\pi}{2}} x\,t\,u_0(t)dt = -2x + \frac{\pi}{2}x + \frac{\pi^3}{12}x - \frac{\pi^4}{48}x\,. \tag{4.80}$$

The noise terms $\mp\frac{\pi}{2}x$ and $\mp 2x$ appear in $u_0(x)$ and $u_1(x)$. Canceling these terms from the zeroth component $u_0(x)$ gives the exact solution

$$u(x) = cos(x) - sin(x), \tag{4.81}$$

that satisfies the integral equation. ■

4.4 The Variational Iteration Method (VIM)

In **Chapter 3**, Sec. 3.7 we presented the *variational iteration method* (VIM) for handling the Volterra integral equations. ***Ji-Huan He*** developed this method, that proved to be reliable in the study of linear and nonlinear, and homogeneous and inhomogeneous equations. Recall that the Adomian decomposition method gives the components of the decomposition series by using a recurrence relation. Unlike the ADM, the variational iteration method gives the successive approximations of the exact solution that may converge to the exact solution in case this solution exists; as we showed in **Chapter 3**. Therefore, one significant feature of this method is that it can handle especially nonlinear problems without the use of the so-called Adomian polynomials as required by the Adomian decomposition method. However, for concrete problems the obtained approximations can be used for numerical reasons.

It is interesting to note that the VIM is used for a differential equation, ordinary or partial, and for an integro-differential equation. This means that to use this method for solving integral equations, we first should convert the integral equation to its equivalent differential equation, or to its equivalent integro-differential equation, by using any appropriate method. Unlike the Adomian method that can be used directly to solve an integral equation, the VIM will be employed to the converted differential or integro differential equation.

In this section, we will apply the variational iteration method to handle Fredholm integral equation. The method works effectively if the kernel $K(x,t)$ is ***separable*** of the form $K(x,t) = g(x)h(t)$. The FIE can be converted to an identical Fredholm integro-differential equation by differentiating both sides, where an initial condition should also be derived. For simplicity, we will study only the cases where $g(x) = x^n, n \geq 1$. In what follows we will present the main steps for using this method. The standard FIE is of the form

$$u(x) = f(x) + \int_a^b K(x,t)\,u(t)dt\,, \tag{4.82}$$

or equivalently

$$u(x) = f(x) + g(x)\int_a^b h(t)\,u(t)dt\,, \quad K(x,t) = g(x)h(t) \qquad (4.83)$$

Recall that the integral at the right side of Eq. (4.83) depends on t only, hence it is equivalent to a constant. Differentiating both sides of Eq. (4.83) with respect to x gives

$$u'(x) = f'(x) + g'(x)\int_a^b h(t)\,u(t)dt\,, \qquad (4.84)$$

The VIM admits the use of a correction functional for the integro-differential equation (4.84) in the form

$$u_{n+1}(x) = u_n(x) + \int_0^x \gamma(\xi)\left(u_n'(\xi) - f'(\xi) - g'(\xi)\int_0^x h(r)\tilde{u}_n(r)\,dr\right)d\xi \quad (4.85)$$

where γ is a general **Lagrange multiplier**. Note that the Lagrange multiplier γ may be a **constant** or a **function**, and $\tilde{u}_n$ is a restricted value that means it behaves as a constant, hence $\delta\tilde{u}_n = 0$, where δ is the **variational derivative**. The Lagrange multiplier γ can be identified optimally via the variational theory as proved in the literature.

The (VIM) depends mainly on **two essential steps**. We first should determine the Lagrange multiplier $\gamma(\xi)$ that can be identified optimally via the variational theory where integration by parts should be used. As stated before, γ may be a **constant** or a **function**, and **it is different from one question to another**. A list of some of these Lagrange multipliers were given **Chapter 3**, but this list does not cover all differential or integral problems. However for our present use in this chapter, $\gamma(\xi) = -1$ for first order integro-differential equations. Having determined γ, an iteration formula, restricted variation, given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - f'(\xi) - g'(\xi)\int_0^x h(r)u_n(r)\,dr\right)d\xi \qquad (4.86)$$

is used to determine the successive approximations $u_{n+1}(x)\,,\ n \geq 0$ of the solution $u(x)$.

Notice that $u_n(x)$ gives the successive approximations of the solution and not the components as in the case when Adomian method is used. In other words, the correction functional will give several approximations of the solution.

The zeroth approximation $u_0(x)$ can be any selective function. However, using the given initial value $u(0)$ is preferably used for the selective zeroth approximation $u_0(x)$ as will be seen later. Consequently, the solution is given by

$$u(x) = \lim_{n\to\infty} u_n(x). \tag{4.87}$$

The VIM will be illustrated by studying the following FIEs.

Example 1: Solve the following FIE by using the VIM

$$u(x) = xe^x - x + x\int_0^1 u(t)\,dt\,. \tag{4.88}$$

Solution: Differentiating both sides of Eq.(4.88) with respect to x, yields

$$u'(x) = xe^x + e^x - 1 + \int_0^1 u(t)\,dt, \qquad u(0) = 0. \tag{4.89}$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - \xi e^{\xi} - e^{\xi} + 1 - \int_0^1 u_n(r)\,dr \right) d\xi \tag{4.90}$$

where we used $\gamma = -1$ for first-order integro-differential equations. It is preferable to select $u_0(x) = u(0) = 0$. Using this selection into the correction functional gives the following successive approximations

$$u_0(x) = 0, \tag{4.91}$$

$$u_1(x) = u_0 - \int_0^x \left(u_0'(\xi) - \xi e^{\xi} - e^{\xi} + 1 - \int_0^1 u_0(r)\, dr \right) d\xi = \boldsymbol{xe^x - x} \qquad (4.92)$$

$$u_2(x) = u_1 - \int_0^x \left(u_1'(\xi) - \xi e^{\xi} - e^{\xi} + 1 - \int_0^1 u_1(r)\, dr \right) d\xi = \boldsymbol{xe^x - \frac{1}{2}x} \qquad (4.93)$$

$$u_3(x) = u_2 - \int_0^x \left(u_2'(\xi) - \xi e^{\xi} - e^{\xi} + 1 - \int_0^1 u_2(r)\, dr \right) d\xi = \boldsymbol{xe^x - \frac{1}{4}x} \qquad (4.94)$$

$$\vdots$$

$$u_n(x) = \boldsymbol{xe^x - \frac{1}{2^{n-1}}x}\,, n \geq 1. \qquad (4.95)$$

The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x) = xe^x. \ \blacksquare \qquad (4.96)$$

Example 2: Use the VIM to solve the following FIE

$$u(x) = cos(x) - x \ + x \int_0^{\frac{\pi}{2}} u(t)\, dt\,. \qquad (4.97)$$

Solution: Differentiating both sides of Eq.(4.97) with respect to x, yields

$$u'(x) = -sin(x) - 1 + \int_0^{\frac{\pi}{2}} u(t)\, dt, \qquad u(0) = 1. \qquad (4.98)$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) + sin(\xi) + 1 - \int_0^{\frac{\pi}{2}} u_n(r)\, dr \right) d\xi \qquad (4.99)$$

where we used $\gamma = -1$ for first-order integro-differential equations. The initial condition $u(0) = 1$ is obtained by using $x = 0$ into Eq.(4.97), and hence we can select $u_0(x) = 1$. Using this selection into the correction functional gives the following successive approximations

$$u_0(x) = 1, \qquad (4.100)$$

$$u_1(x) = u_0 - \int_0^x \left(u_0'(\xi) + sin(\xi) + 1 - \int_0^{\frac{\pi}{2}} u_0(r)\, dr \right) d\xi = \boldsymbol{cos(x) - x + \frac{\pi}{2} x} \qquad (4.101)$$

$$u_2(x) = u_1 - \int_0^x \left(u_1'(\xi) + sin(\xi) + 1 - \int_0^{\frac{\pi}{2}} u_1(r)\, dr \right) d\xi$$

$$= \boldsymbol{(cos(x) - x) + \left(x - \frac{\pi^2}{8} x \right) + \frac{\pi^3}{16} x} \qquad (4.102)$$

$$u_3(x) = u_2 - \int_0^x \left(u_2'(\xi) + sin(\xi) + 1 - \int_0^{\frac{\pi}{2}} u_2(r)\, dr \right) d\xi$$

$$= \boldsymbol{(cos(x) - x) + \left(x - \frac{\pi^2}{8} x \right) + \left(\frac{\pi^2}{8} x - \frac{\pi^3}{16} x \right) + \left(\frac{\pi^3}{16} x - \cdots \right) + \cdots} \qquad (4.103)$$

and so on. Canceling the noise terms, the exact solution is given by

$$u(x) = cos(x). \ \blacksquare \qquad (4.104)$$

4.5 The Direct Computation Method

In this section we introduce an efficient traditional method for solving Fredholm integral equations of the second kind (4.1), and this method is usually known as the *direct computational method*. In this method, the kernel $K(x,t)$ ***must be separable*** or ***degenerate*** in the product form such that it can be expressed as in Eq. (4.4). Without loss of generality, we may assume that the kernel of Eq.(4.1) can be expressed as

$$K(x,t) = g(x)h(t)\,. \tag{4.105}$$

Accordingly, Eq. (4.1) can be expressed as

$$u(x) = f(x) + \lambda\, g(x) \int_a^b h(t)\, u(t)dt\,. \tag{4.106}$$

The integral equation (4.106) is a ***definite integral*** and the integral at the right hand side of this equation reveals that the integrand depends on one variable, namely the variable t . This means that the definite integral in the right hand side of Eq. (4.106) is equivalent to a numerical value α, where α is an unknown constant that needs to be determined. In other words, we may write

$$\alpha = \int_a^b h(t)\, u(t)dt\,. \tag{4.107}$$

It follows that Eq. (4.106) can be written as

$$u(x) = f(x) + \lambda\, \alpha\, g(x). \tag{4.108}$$

It is thus obvious that the solution $u(x)$ is completely determined provided α is known from Eq. (4.107). This can be easily done by substitution Eq. (4.108) into Eq. (4.107). We point out here that this approach is slightly different than other existing techniques in that we substitute Eq. (4.108) into Eq. (4.107) and not in Eq. (4.106) as used by other texts.

Remark 4.6: It is worth noting that the *direct computation method* determines the exact solution in a closed form, rather than a series form, provided that the constant α has been determined. In addition, this method usually gives rise to a system of algebraic equations

depending on the structure of the kernel $K(x,t)$, where sometimes we need to evaluate more than one constant as will be seen in **Examples 3 and 4**. The computational difficulties may arise in determining the constant α if the resulting algebraic equation is of third order or higher. This kind of difficulty may arise in the nonlinear integral equation. For linear Fredholm integral equations, we obtain one value for α, or one value for each of α and β if these two constants are used. This is due to the fact that linear equation has a unique solution.

In what follows, we examine four illustrative examples by using the direct computation method.

Example 1: Solve the following FIE by using the direct computational method

$$u(x) = \frac{5}{6}x + \frac{1}{2}\int_0^1 x\, t\, u(t)\, dt\,. \tag{4.109}$$

Solution: As indicated before we set

$$\alpha = \int_0^1 t\, u(t)\, dt\,. \tag{4.110}$$

where α is a constant that represents the numerical value of the integral (4.110). The Eq. (4.110) carries Eq. (4.109) into

$$u(x) = \left(\frac{5}{6} - \frac{1}{2}\alpha\right)x\,. \tag{4.111}$$

To determine α, we substitute Eq. (4.111) into Eq. (4.110) to obtain

$$\alpha = \int_0^1 \left(\frac{5}{6} - \frac{1}{2}\alpha\right)t^2\, dt\,. \tag{4.112}$$

so that by integrating the right hand side and solving for α we find

$$\alpha = \frac{1}{3}\,. \tag{4.113}$$

Substituting Eq. (4.113) into Eq. (4.111) yields

$$u(x) = x\,. \tag{4.114}$$

the exact solution of the given FIE. ■

Example 2: Solve the following FIE by using the direct computational method

$$u(x) = sec^2(x) - 1 + \int_0^{\frac{\pi}{4}} u(t)\,dt\,. \tag{4.115}$$

Solution: As indicated before we set

$$\alpha = \int_0^{\frac{\pi}{4}} u(t)\,dt\,. \tag{4.116}$$

where α is a constant that represents the numerical value of the integral (4.115). By substituting Eq. (4.116) into Eq. (4.115) yields

$$u(x) = sec^2(x) - 1 + \alpha\,. \tag{4.117}$$

To determine α, we substitute Eq. (4.117) into Eq. (4.116) to obtain

$$\alpha = \int_0^{\frac{\pi}{4}} (sec^2(x) - 1 + \alpha)\,dt\,. \tag{4.118}$$

so that by integrating the right hand side and solving for α we find

$$\alpha = 1\,. \tag{4.119}$$

Substituting Eq. (4.119) into Eq. (4.117) yields

$$u(x) = sec^2(x)\,. \tag{4.120}$$

the exact solution of the given FIE. ■

Example 3: Solve the following FIE by using the direct computational method

$$u(x) = -8x - 6x^2 + \int_0^1 (20xt^2 + 12x^2t)\, u(t)dt\,. \qquad (4.121)$$

Solution: Noting that the kernel here is separable and consists of two terms, we can rewrite Eq. (4.121) as

$$u(x) = -8x - 6x^2 + 20x \int_0^1 t^2\, u(t)dt + 12x^2 \int_0^1 t\, u(t)dt. \qquad (4.122)$$

Now, we set

$$\alpha = \int_0^1 t^2\, u(t)dt\,, \qquad (4.123)$$

and

$$\beta = \int_0^1 t\, u(t)dt\,. \qquad (4.124)$$

where α and β are constants. Consequently, Eq. (4.122) can be expressed in the form

$$u(x) = (20\alpha - 8)x + (12\beta - 6)x^2. \qquad (4.125)$$

Substituting Eq. (4.125) into Eq. (4.123) and Eq. (4.124) we obtain

$$\alpha = \int_0^1 [(20\alpha - 8)t + (12\beta - 6)t^2]\, t^2 dt\,, \qquad (4.126)$$

and

$$\beta = \int_0^1 [(20\alpha - 8)t + (12\beta - 6)t^2]\, t dt\,, \qquad (4.127)$$

Integrating the right hand side of equations (4.126) and (4.127) yields the system of equations

$$5\alpha + 3\beta = 4\,, \qquad (4.128)$$

$$40\alpha + 12\beta = 25\,, \qquad (4.129)$$

so that by solving this system we find

$$\alpha = \frac{9}{10}\,, \qquad \beta = \frac{7}{12}\,. \qquad (4.130)$$

Inserting Eq. (4.130) into Eq. (4.125) gives

$$u(x) = x^2 + x\,. \ \blacksquare \qquad (4.131)$$

Example 4: Solve the following FIE by using the direct computational method

$$u(x) = 1 + 9x + 2x^2 + x^3 - \int_0^1 (20xt + 10x^2t^2)\, u(t)dt\,. \qquad (4.132)$$

Solution: Noting that the kernel here is separable and consists of two terms, we can rewrite Eq. (4.132) as

$$u(x) = 1 + 9x + 2x^2 + x^3 - 20x \int_0^1 t\, u(t)dt + 12x^2 \int_0^1 t^2\, u(t)dt. \qquad (4.133)$$

Now, we set

$$\alpha = \int_0^1 t\, u(t)dt\,, \qquad (4.134)$$

and

$$\beta = \int_0^1 t^2\, u(t)dt\,. \qquad (4.135)$$

where α and β are constants. Consequently, Eq. (4.133) can be expressed in the form

$$u(x) = 1 + (9 - 20\alpha)x + (2 - 10\beta)x^2 + x^3. \qquad (4.136)$$

Substituting Eq. (4.136) into Eq. (4.134) and Eq. (4.135) we obtain

$$\alpha = \int_0^1 [1 + (9 - 20\alpha)t + (2 - 10\beta)t^2 + t^3]\, tdt\,, \qquad (4.137)$$

and

$$\beta = \int_0^1 [1 + (9 - 20\alpha)t + (2 - 10\beta)t^2 + t^3]\, t^2dt\,, \qquad (4.138)$$

Integrating the right hand side of equations (4.137) and (4.138) yields the system of equations

$$230\alpha + 75\beta = 126\,, \qquad (4.139)$$

$$100\alpha + 60\beta = 63\,, \qquad (4.140)$$

so that by solving this system we find

$$\alpha = \frac{9}{20}\ , \qquad \beta = \frac{3}{10}\,. \qquad (4.141)$$

Inserting Eq. (4.141) into Eq. (4.136) gives

$$u(x) = 1 - x^2 + x^3\,. \ \blacksquare \qquad (4.142)$$

In closing this section, we point out that the direct computation method introduces a very direct technique to formally determine the solution of FIE. In this method, the FIE will be transformed into a more readily solvable integral. Moreover, the direct computation method introduces the exact solution in a closed form rather than a series form as in the case of the

decomposition method or the VIM. The other traditional methods, that will be discussed in the forthcoming sections, also determine the solution in a series form, but in a different approach than the decomposition method or the VIM.

4.6 The method of successive approximations: Neumann's series

The successive approximation method, which was successfully applied to Volterra integral equations of the second kind, can be applied even more easily to the basic Fredholm integral equations (FIEs) of the second kind:

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)\,dt, \qquad a \le x \le b \tag{4.143}$$

In this method, we replace the unknown function under the integral sign of the FIE by any selective real-valued function $u_0(x)$, $a \le x \le b$. We set $u_0(x) = f(x)$. Note that, $u_0(x)$ is the zeroth approximation. Accordingly, the first approximation $u_1(x)$ of the solution of $u(x)$ is defined by

$$u_1(x) = f(x) + \lambda \int_a^b K(x,t)u_0(t)\,dt\,. \tag{4.144}$$

The second approximation $u_2(x)$ of the solution $u(x)$ can be obtained by replacing $u_0(x)$ in Eq. (4.144) by the previously obtained $u_1(x)$; hence we find

$$u_2(x) = f(x) + \lambda \int_a^b K(x,t)u_1(t)\,dt\,. \tag{4.145}$$

This process can be continued in the same manner to obtain the nth approximation. In other words, the various approximations can be put in a recursive scheme given by

$$u_0(x) = \text{any selective real-valued function}$$

$$u_n(x) = f(x) + \lambda \int_a^b K(x,t)u_{n-1}(t)\,dt\,, n \geq 1. \tag{4.146}$$

Even though we can select any real-valued function for the zeroth approximation $u_0(x)$, the most commonly selected functions for $u_0(x)$ are $u_0(x) = 0, 1,$ or x. We have noticed that with the selection of $u_0(x) = 0,$ the first approximation $u_1(x) = f(x).$ The final solution $u(x),$ is obtained by

$$u(x) = \lim_{n\to\infty} u_n(x), \tag{4.147}$$

so that the resulting solution $u(x)$ is **<u>independent</u>** of the choice of $u_0(x)$. This is known as ***Picard's method***.

The Neumann series is obtained if we set $u_0(x) = f(x)$ such that

$$u_1(x) = f(x) + \lambda \int_a^b K(x,t)u_0(t)\,dt \tag{4.148}$$

$$= f(x) + \lambda \int_a^b K(x,t)f(t)\,dt \tag{4.149}$$

$$= f(x) + \lambda\, \psi_1(x). \tag{4.150}$$

where

$$\psi_1(x) = \int_a^b K(x,t)f(t)\,dt \tag{4.151}$$

The second approximation $u_2(x)$ can be obtained as

$$u_2(x) = f(x) + \lambda \int_a^b K(x,t)u_1(t)\,dt \tag{4.152}$$

$$= f(x)\lambda + \int_a^b K(x,t)\{f(t) + \lambda\,\psi_1(t)\}\,dt \qquad (4.153)$$

$$= f(x) + \lambda\,\psi_1(x) + \lambda^2\,\psi_2(x). \qquad (4.154)$$

where

$$\psi_2(x) = \int_a^b K(x,t)\psi_1(t)\,dt \qquad (4.155)$$

Proceeding in this manner, the final solution $u(x)$ can be obtained

$$u(x) = f(x) + \lambda\,\psi_1(x) + \lambda^2\,\psi_2(x) + \cdots + \lambda^n\,\psi_n(x) + \cdots \qquad (4.156)$$

$$= f(x) + \sum_{n=1}^{\infty} \lambda^n\,\psi_n(x)\,, \qquad (4.157)$$

where

$$\psi_n(x) = \int_a^b K(x,t)\psi_{n-1}(t)\,dt, \qquad n \geq 1 \qquad (4.158)$$

Series (4.157) is usually known as **Neumann series**. This infinite series can be shown to be absolutely and uniformly convergent.

Remark 4.7: It can be later seen that the Neumann series is identical with the Adomian decomposition method for the linear Fredholm integral equation. And the successive approximation is identical with the Picard's method. For ready reference we cite below:

Picard's method:

$$u(x) = \lim_{n\to\infty} u_n(x), \qquad (4.159)$$

Neumann's series method:

$$u(x) = f(x) + \lim_{n\to\infty} \sum_{k=1}^{n} \lambda^k \psi_k(x) \,, \tag{4.160}$$

Remark 4.8: It is important to distinguish between the recursive schemes used in the Adomian decomposition method and in the successive approximations method. In the decomposition method, we apply the approach to determine several components of the solution $u(x)$ where, in this case

$$u(x) = \sum_{n=0}^{\infty} u_n\,(x) \,, \tag{4.161}$$

so that the zeroth component $u_0(x)$ is defined by all terms that are out of the integral sign or part of these terms if the modified version is used. However, in the successive approximations method, we apply the above recursive scheme Eq.(4.146) to determine various approximations of the solution $u(x)$ itself, and not components of $u(x)$. Further, we should note here that the zeroth approximation $u_0(x)$ is not defined but rather given by a selective function, and as a result the solution $u(x)$ is given by the formula (4.159). The successive approximations method will be illustrated by the following examples.

Example 1: Solve the FIE by using the successive approximation method

$$u(x) = e^x + e^{-1} \int_0^1 u(t)\, dt \,. \tag{4.162}$$

Solution: As indicated above we can select any real-value function for the zeroth component, hence we set

$$u_0(x) = 0 \,. \tag{4.163}$$

Substituting Eq. (4.163) into the right hand side of Eq. (4.162) we find

$$u_1(x) = e^x + e^{-1} \int_0^1 u_0(t)\, dt \,. \tag{4.164}$$

and this gives the first approximation of $u(x)$ by

$$u_1(x) = e^x \,. \tag{4.165}$$

Inserting Eq. (4.165) into Eq. (164) to replace $u_0(x)$ we obtain

$$u_2(x) = e^x + e^{-1}\int_0^1 e^t\,dt \,. \tag{4.166}$$

where by integration we determine the second approximation of $u(x)$ by

$$u_2(x) = e^x + 1 - e^{-1}. \tag{4.167}$$

Continuing in the same manner we find the third approximation of $u(x)$ given by

$$u_3(x) = e^x + 1 - e^{-2}. \tag{4.168}$$

Proceeding as before, we obtain the *n*th component

$$u_n(x) = e^x + 1 - e^{-(n-1)}\,,\ \ n \geq 1. \tag{4.169}$$

Using Eq. (4.159), the solution $u(x)$ of Eq. (4.162) is given by

$$u(x) = \lim_{n\to\infty} u_n(x), \tag{4.170}$$

$$= \lim_{n\to\infty}\left(e^x + 1 - e^{-(n-1)}\right), \tag{4.171}$$

$$= e^x + 1\,, \tag{4.172}$$

obtained upon evaluating the limit as $n \to \infty$. ■

Example 2: Solve the FIE by using the successive approximation method

$$u(x) = x + \lambda\int_0^1 xtu(t)\,dt \,. \tag{4.173}$$

Solution: As indicated above we can select any real-value function for the zeroth component, hence we set

$$u_0(x) = 0\,. \tag{4.174}$$

Substituting Eq. (4.174) into the right hand side of Eq. (4.172), the first approximation of $u(x)$ is given by

$$u_1(x) = x. \tag{4.175}$$

Follows immediately. Proceeding in the same manner we find that

$$u_2(x) = x + \lambda \int_0^1 xt^2\,dt\,. \tag{4.176}$$

where by integration we determine the second approximation of $u(x)$ by

$$u_2(x) = x + \frac{\lambda}{3}x. \tag{4.177}$$

Continuing in the same manner we find the third approximation of $u(x)$ given by

$$u_3(x) = x + \lambda \int_0^1 xt\left(1 + \frac{\lambda}{3}\right)tdt. \tag{4.178}$$

$$= x + \frac{\lambda}{3}x + \frac{\lambda^2}{9}x\,. \tag{4.179}$$

Proceeding as before, we obtain the *n*th component

$$u_n(x) = x + \frac{\lambda}{3}x + \frac{\lambda^2}{9}x + \cdots + \frac{\lambda^{n-1}}{3^{n-1}}x\,,\ \ n \geq 1. \tag{4.180}$$

Consequently, the solution *u*(*x*) of Eq. (4.173) is given by

$$u(x) = \lim_{n\to\infty} u_n(x), \tag{4.181}$$

$$= \lim_{n\to\infty}\left(x + \frac{\lambda}{3}x + \frac{\lambda^2}{9}x + \cdots + \frac{\lambda^{n-1}}{3^{n-1}}x\right), \tag{4.182}$$

$$= \frac{3}{3-\lambda}x\,, \qquad 0 < \lambda < 3\,. \tag{4.183}$$

obtained upon evaluating the limit as $n \to \infty$. ■

To show that $u(x)$ obtained in Eq. (4.183) does not depend on the selection of $u_0(x)$, we will solve the equation Eq.(4.173) by selecting

$$u_0(x) = x\,. \tag{4.184}$$

Using the new selection of $u_0(x)$ in the right hand side of Eq.(4.173), the first approximation

$$u_1(x) = x + \frac{\lambda}{3}x\,. \tag{4.185}$$

is readily obtained. Proceeding as before we thus obtain

$$u_2(x) = x + \lambda\int_0^1 xt\left(t + \frac{\lambda}{3}t\right)dt\,. \tag{4.186}$$

which gives

$$u_2(x) = x + \frac{\lambda}{3}x + \frac{\lambda^2}{9}x\,. \tag{4.187}$$

In a parallel manner we find

$$u_n(x) = x + \frac{\lambda}{3}x + \frac{\lambda^2}{3^2}x + \cdots + \frac{\lambda^n}{3^n}x\,, \;\; n \geq 1. \tag{4.188}$$

Accordingly, we obtain

$$u(x) = \frac{3}{3-\lambda} x\,, \qquad 0 < \lambda < 3\,. \tag{4.189}$$

which is consistent with the same result obtained above in Eq. (183). ■

4.7 The method of successive Substitutions

The method of successive substitutions, that will be used here, is completely identical to that used in **Chapter 3** for handling Volterra integral equations. This method is almost analogous to the successive approximation method except that it concerns with the solution of the integral equation in a series form through evaluating single and multiple integrals as well. The solution by the numerical procedure in this method is huge compared to other techniques. In this method, we set $x = t$ and $t = t_1$ in the Fredholm integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)\,dt, \qquad a \le x \le b\,. \tag{4.190}$$

to obtain

$$u(t) = f(t) + \lambda \int_a^b K(t,t_1)u(t_1)\,dt_1, \qquad a \le x \le b\,. \tag{4.191}$$

Replacing $u(t)$ in the right hand side of Eq. (4.190) by its obtained value given by Eq. (4.191) yields

$$u(x) = f(x) + \lambda \int_a^a K(x,t)\left[f(t) + \lambda \int_a^b K(t,t_1)u(t_1)dt_1\right]dt, \tag{4.192}$$

$$= f(x) + \lambda \int_a^b K(x,t)\,f(t)dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1)u(t_1)dt_1\,dt. \tag{4.193}$$

Substituting $x = t_1$ and $t = t_2$ in Eq.(4.190) we obtain

$$u(t_1) = f(t_1) + \lambda \int_a^b K(t_1, t_2) u(t_2)\, dt_2. \qquad (4.194)$$

Substituting the value of $u(t_1)$ obtained in Eq.(4.194) into the right hand side of Eq.(4.193) leads to

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, f(t) dt$$

$$+\lambda^2 \int_a^b \int_a^b K(x,t) K(t,t_1) f(t_1) dt_1 dt$$

$$+\lambda^3 \int_a^b \int_a^b \int_a^b K(x,t) K(t,t_1)\, K(t_1,t_2) u(t_2) dt_2 dt_1 dt$$

$$+ \cdots \qquad (4.195)$$

Accordingly, the general series form for $u(x)$ can be rewritten as

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, f(t) dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1) f(t_1) dt_1\, dt + \cdots$$
$$+ \lambda^n \int_a^b K(x,t) \int_a^b K(t,t_1) \ldots \times \int_a^b K(t_{n-2}, t_{n-1}) f(t_{n-1}) dt_{n-1} \ldots dt_1 dt$$
$$+ R_{n+1}(x), \qquad (4.196)$$

where

$$R_{n+1}(x) = \lambda^{n+1} \int_a^b K(x,t) \int_a^b K(t,t_1) \ldots \int_a^b K(t_{n-1}, t_n) u(t_n) dt_n \ldots dt_1 dt \ \ldots \qquad (4.197)$$

is the remainder after *n* terms. It can be shown that $\lim_{n\to\infty} R_{n+1} = 0$.

We note that the series solution given in Eq. (4.196) converges uniformly in the interval $[a, b]$ if $\lambda M(b-a) < 1$ where $|K(x,t)| \leq M$. We remark here that in this method the

unknown function $u(x)$ is substituted by the given function $f(x)$ that makes the evaluation of the multiple integrals easily computable. This substitution of $u(x)$ occurs several times through the integrals and **hence this is why it is called the method of successive substitutions**. The technique will be illustrated by discussing the following examples.

Example 1: Solve the following FIE by using the method of successive substitutions

$$u(x) = \frac{23}{6}x + \frac{1}{8}\int_0^1 xtu(t)\,dt, \tag{4.198}$$

Solution: Substituting $\lambda = \frac{1}{8}$, $f(x) = \frac{23}{6}x$, and $K(x,t) = xt$ into Eq.(4.195) yields

$$u(x) = \frac{23}{6}x + \frac{1}{8}\int_0^1 \frac{23}{6}xt^2u(t)\,dt + \frac{1}{8^2}\int_0^1\int_0^1 \frac{23}{6}xt_1^2t^2dt_1dt + \cdots, \tag{4.199}$$

or equivalently

$$u(x) = \frac{23}{6}x\left[1 + \frac{1}{24} + \frac{1}{576} + \cdots\right], \tag{4.200}$$

so that we obtain the solution

$$u(x) = 4x \tag{4.201}$$

upon evaluating the sum of the geometric series. ■

Example 2: Solve the following FIE by using the method of successive substitutions

$$u(x) = 1 + \frac{1}{4}\int_0^{\frac{\pi}{2}} cos(x)u(t)\,dt, \tag{4.202}$$

Solution: Substituting $\lambda = \frac{1}{4}$, $f(x) = 1$, and $K(x,t) = cos(x)$ into Eq.(4.195) yields

$$u(x) = 1 + \frac{1}{4}\int_0^{\frac{\pi}{2}} cos(x)\,dt + \frac{1}{16}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} cos(x)cos(t)dt_1dt + \cdots \tag{4.203}$$

and this will yield

$$u(x) = 1 + \left(\frac{\pi}{8}cos(x) + \frac{\pi}{32}cos(x) + \cdots\right) \tag{4.204}$$

Consequently we easily obtain the exact solution

$$u(x) = 1 + \frac{\pi}{6}cos(x)\,, \tag{4.205}$$

obtained upon using the sum of the infinite geometric series. ■

4.8 Homogeneous Fredholm Equations

In this section we will study the homogeneous Fredholm integral equation with separable kernel given by

$$u(x) = \lambda \int_a^b K(x,t)\,u(t)dt, \tag{4.206}$$

obtained from Eq.(4.1) by setting $f(x) = 0$.

It is easily seen that the trivial solution $u(x) = 0$ is a solution of the homogeneous Fredholm integral equation (4.206). In this study our goal will be focused on finding **nontrivial** solutions to Eq. (4.206) if exist. We can achieve our goal by introducing the technique that will enable us to determine the **nontrivial** solutions to Eq.(4.206).

Generally speaking, the homogeneous FIE with separable kernel may have nontrivial solutions. Our approach in obtaining these desired solutions will be based mainly on the *direct computation method* that was employed effectively for nonhomogeneous FIE.

We point out that ADM **is not applicable** for the homogeneous FIEs. This may be related to the fact that the nonhomogeneous part $f(x)$ does not exist in this type of problems, and therefore the zeroth component $u_0(x)$ cannot be defined.

We recall that the direct computation method reduces the equation to an algebraic equation if the kernel consists of one term only, or to a system of algebraic equations if the kernel contains many separable terms. Additional discussions will be required for determining possible values of λ that will give rise to the nontrivial solutions as will be discussed soon.

Without loss of generality we may assume a one term kernel given by

$$K(x,t) = g(x)h(t) \tag{4.207}$$

so that Eq. (4.206) becomes

$$u(x) = \lambda g(x) \int_a^b h(t)\, u(t)dt, \tag{4.208}$$

Now, using the direct computation method we set

$$\alpha = \int_a^b h(t)\, u(t)dt, \tag{4.209}$$

so that Eq. (4.208) becomes

$$u(x) = \lambda \alpha g(x)\,. \tag{4.210}$$

We note that $\alpha = 0$ gives the trivial solution $u(x) = 0$, by using Eq. (4.210), which is not our desired goal in this study. However, to determine the nontrivial solutions of Eq.(4.206), we need to determine the values of λ by considering $\alpha \neq 0$. This can be done by substituting Eq. (4.210) into Eq. (4.209) to obtain

$$\alpha = \lambda \alpha \int_a^b h(t)\, u(t)dt, \tag{4.211}$$

or equivalently

$$1 = \lambda \int_a^b h(t)\, u(t)dt, \tag{4.212}$$

which gives a numerical value for $\lambda \neq 0$, by evaluating the definite integral in Eq. (4.212). Having evaluated λ, the nontrivial solution given by Eq. (4.210) is determined.

For separable kernels that contain more than one term, the method reduces the homogeneous Fredholm integral equation to a system of algebraic equations as will be seen from the examples below.

In closing this section we point out that the particular nonzero values of λ that result from solving the algebraic system of equations are called the ***eigenvalues*** of the kernel. Moreover, substituting the obtained values of λ in Eq. (4.210) gives the usually called ***eigensolutions*** or ***eigenfunctions*** of the equation which are the nontrivial solutions of Eq. (4.206) within a finite interval (a, b).

The following illustrative examples will be used to explain the technique introduced above and the concept of *eigenvalues* and *eigenfunctions*.

Example 1: Solve the homogeneous Fredholm integral equation with one term kernel

$$u(x) = \lambda \int_0^1 x\, u(t) dt, \tag{4.213}$$

Solution: As indicated above, Eq. (4.213) becomes

$$u(x) = \lambda \alpha x, \tag{4.214}$$

where

$$\alpha = \int_0^1 u(t)\, dt, \tag{4.215}$$

Substituting Eq. (4.214) into Eq. (4.215) yields

$$\alpha = \lambda \alpha \int_0^1 t\, dt, \tag{4.216}$$

which gives

$$\alpha = \frac{1}{2} \lambda \alpha, \tag{4.217}$$

so that the eigenvalue of the kernel

$$\lambda = 2\,, \tag{4.218}$$

obtained by noting that $\alpha \neq 0$. Substituting Eq. (4.218) into Eq. (4.217) yields

$$\alpha = \alpha\,, \tag{4.219}$$

which indicates that α is an arbitrary constant. Using Eq. (4.218) and Eq.(4.219) in Eq. (4.214) leads to the eigenfunction of the equation given by

$$u(x) = 2\alpha x, \tag{4.220}$$

obtained upon using Eq. (4.214). ■

Example 2: Find the nontrivial solutions of the following Fredholm integral equation

$$u(x) = \frac{2}{\pi}\lambda \int_0^{\pi} cos(x+t)\, u(t)dt, \tag{4.221}$$

Solution: The equation (4.221) can be rewritten as

$$u(x) = \frac{2}{\pi}\lambda\, cos(x) \int_0^{\pi} cos(t)\, u(t)dt - \frac{2}{\pi}\lambda\, sin(x) \int_0^{\pi} sin(t)\, u(t)dt, \tag{4.222}$$

or by

$$u(x) = \frac{2}{\pi}\lambda\, [\alpha cos(x) - \beta sin(x)], \tag{4.223}$$

where

$$\alpha = \int_0^{\pi} cos(t)\, u(t)dt, \tag{4.224}$$

and

$$\beta = \int_0^{\pi} sin(t)\, u(t)dt, \tag{4.225}$$

Substituting Eq. (4.223) into Eq. (4.224) and Eq. (4.225) and integrating yield

$$\alpha = \lambda\alpha, \tag{4.226 }$$

and

$$\beta = -\lambda\beta, \tag{4.227}$$

For $\alpha \neq 0$ and $\beta \neq 0$, we obtain the eigenvalues

$$\lambda_1 = 1 \ and \ \lambda_2 = -1 \tag{4.228}$$

Substituting $\lambda_1 = 1$ in Eq. (4.226) and Eq. (4.227) yields

$$\alpha = \alpha\,, \qquad \beta = 0\,, \tag{4.229}$$

which gives the eigenfunction corresponding to $\lambda_1 = 1$ by

$$u_1(x) = \frac{2}{\pi}\alpha cos(x)\,, \tag{4.230}$$

obtained upon using Eq. (4.223), where α is an arbitrary constant.

Similarly substituting $\lambda_2 = -1$ in Eq. (4.226) and Eq. (4.227) yields

$$\alpha = 0\,, \qquad \beta = \beta\,, \tag{4.231}$$

which gives the eigenfunction corresponding to $\lambda_2 = -1$ by

$$u_2(x) = \frac{2}{\pi}\beta sin(x)\,, \tag{4.232}$$

obtained upon using Eq. (4.223), where β is an arbitrary constant. ■

4.9 Fredholm Integral Equations of the First Kind

It was stated before that in Fredholm integral equations of the first kind, the unknown function $u(x)$ appears only inside the integral sign. The standard form of the Fredholm equations of the first kind reads

$$f(x) = \lambda \int_a^b K(x,t)\, u(t)dt, \qquad x \in D, \tag{4.233}$$

where $f(x)$ is the data. The occurrence of $u(x)$ under the integral sign makes it difficult to apply the aforementioned method. It was proved in the literature that the data function $f(x)$ must lie in the range of the kernel $K(x,t)$ This means that if we set the kernel by $K(x,t) = e^x cos(t)$, then for any $u(x)$, the resulting data $f(x)$ must be a multiple of e^x, otherwise the solution $u(x)$ does not exist. The Fredholm integral equations of the first kind appear in many physical models such as radiography, spectroscopy, cosmic radiation, image processing and in the theory of signal processing.

In the literature, the Fredholm integral equations of the first kind is considered **ill posed problem**, and this indicates that this first kind Fredholm equation may have no solution, or if a solution exists it is not unique and may not depend continuously on the data.

There are many methods that were used to investigate the Fredholm integral equations of the first kind, analytically and numerically. However, in this text we will use the ***method of regularization*** that transforms first kind equation to second kind equation. Having converted the equation from the first kind to the second kind enables us to apply any appropriate method that were presented in this chapter.

► 4.9.1 The Method of Regularization

The method of regularization converts the linear Fredholm integral equation of the first kind

$$f(x) = \int_a^b K(x,t)\, u(t)dt, \qquad x \in D, \tag{4.234}$$

to the Fredholm integral equation of the second kind in the form

$$\varepsilon u_\varepsilon(x) = f(x) - \int_a^b K(x,t)\, u_\varepsilon(t)dt, \tag{4.235}$$

or equivalently

$$u_\varepsilon(x) = \frac{1}{\varepsilon} f(x) - \frac{1}{\varepsilon} \int_a^b K(x,t)\, u_\varepsilon(t) dt, \tag{4.236}$$

where ε is a small positive parameter. It is obvious that the solution u of Eq. (4.236) converges to the solution $u(x)$ of Eq. (4.234) as $\varepsilon \to 0$.

In what follows we will present two illustrative examples. Our focus will be on transforming the first kind equation to a second kind equation by using the method of regularization, and hence we can use any appropriate method.

Example 1: Combine the method of *regularization* and the *direct computation method* to solve the Fredholm integral equation of the first kind

$$\frac{1}{2} e^{2x} = \int_0^{\frac{1}{2}} e^{2x-2t}\, u(t) dt\,. \tag{4.237}$$

Solution: Using the method of regularization, Eq. (4.237) becomes

$$u_\varepsilon(x) = \frac{1}{2\varepsilon} e^{2x} - \frac{1}{\varepsilon} \int_0^{\frac{1}{2}} e^{2x-2t}\, u_\varepsilon(t) dt, \tag{4.238}$$

Using the direct computation method, Eq. (4.238) becomes

$$u_\varepsilon(x) = \left(\frac{1}{2\varepsilon} - \frac{\alpha}{\varepsilon} \right) e^{2x}\,, \tag{4.239}$$

where

$$\alpha = \int_0^{\frac{1}{2}} e^{-2t}\, u_\varepsilon(t) dt, \tag{4.240}$$

To determine α, we substitute Eq. (4.239) into Eq. (4.240), integrate and solving to find that

$$\alpha = \frac{1}{2+4\varepsilon}\,. \tag{4.241}$$

This in turn gives

$$u_\varepsilon(x) = \frac{e^{2x}}{1+2\varepsilon}\ . \tag{4.242}$$

The exact solution is given by

$$u(x) = \lim_{\varepsilon\to 0} u_\varepsilon(x) = e^{2x}\ .\ \blacksquare \tag{4.243}$$

Example 2: Combine the method of regularization and the Adomian decomposition method to solve the Fredholm integral equation of the first kind

$$\frac{1}{5}x = \int_0^1 xt\,u(t)dt\,. \tag{4.244}$$

Solution: Using the method of regularization, Eq. (4.244) can be transformed to

$$u_\varepsilon(x) = \frac{1}{5\varepsilon}x - \frac{1}{\varepsilon}\int_0^1 xt\,u_\varepsilon(t)dt, \tag{4.245}$$

To use the ADM, we first select $u_{\varepsilon_0}(x) = 0$. Consequently, we obtain the following approximations

$$u_{\varepsilon_0}(x) = \frac{1}{5\varepsilon}x\,, \tag{4.246}$$

$$u_{\varepsilon_1}(x) = -\frac{1}{15\varepsilon^2}x\,, \tag{4.247}$$

$$u_{\varepsilon_2}(x) = \frac{1}{45\varepsilon^3}x\,, \tag{4.248}$$

$$u_{\varepsilon_3}(x) = -\frac{1}{135\varepsilon^4}x\,, \tag{4.249}$$

$\vdots$

and so on. Based on this we obtain the solution

$$u_\varepsilon(x) = \frac{3}{5(1+3\varepsilon)}x\,. \tag{4.250}$$

The exact solution $u(x)$ of Eq. (4.244) can be obtained by

$$u(x) = \lim_{\varepsilon\to 0} u_\varepsilon(x) = \frac{3}{5}x\ . \tag{4.251}$$

It is interesting to point out that another solution to this equation is given by

$$u(x) = x^3\ . \tag{4.252}$$

This is normal to get more than one solution because the Fredholm integral equation of the first kind is **ill-posed problem**. ■

Chapter 5

Integro-differential equations

5.1 Introduction

This chapter deals with one of the most applied problems in the engineering sciences. It is concerned with the ***integro-differential equations*** where both differential and integral operators will appear in the same equation. This type of equations was introduced by Volterra for the first time in the early 1900. Volterra investigated the population growth, focusing his study on the hereditary influences, where through his research work the topic of integro-differential equations was established (see Abramowitz and Stegun **[1]**).

Scientists and engineers come across the integro-differential equations through their research work in heat and mass diffusion processes, electric circuit problems, neutron diffusion, and biological species coexisting together with increasing and decreasing rates of generating. Applications of the integro-differential equations in electromagnetic theory and dispersive waves and ocean circulations are enormous. More details about the sources where these equations arise can be found in physics, biology, and engineering applications as well as in advanced integral equations literatures. In the electrical *LRC* circuit, one encounters an integro-differential equation to determine the instantaneous current flowing through the circuit with a given voltage $E(t)$, and it is written as

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I(\tau)\,d\tau = E(t) \qquad (A)$$

with the initial condition $I(0) = I_0$ at $t = 0$. To solve this problem we need to know the appropriate technique (see Rahman **[2]**).

It is important to note that in the integro-differential equations, the unknown function $u(x)$ and one or more of its derivatives such as $u'(x), u''(x)$, ... appear out and under the integral

sign as well. One quick source of integro-differential equations can be clearly seen when we convert the differential equation to an integral equation by using Leibnitz rule. The integro-differential equation can be viewed in this case as an intermediate stage when finding an equivalent Volterra integral equation to the given differential equation.

The following are the examples of linear integro-differential equations:

$$u'(x) = f(x) - \int_0^x (x-t)\,u(t)dt\,, \qquad u(0) = 0, \tag{5.1}$$

$$u''(x) = g(x) + \int_0^x (x-t)\,u(t)dt\,, \qquad u(0) = 0, u'(0) = -1, \tag{5.2}$$

$$u'(x) = e^x - x + \int_0^1 xt\,u(t)dt\,, \qquad u(0) = 0, \tag{5.3}$$

$$u''(x) = h(x) + \int_0^1 t\,u'(t)dt\,, \qquad u(0) = 0, u'(0) = 1, \tag{5.4}$$

It is clear from the above examples that the unknown function $u(x)$ or one of its derivatives appear under the integral sign, and the other derivatives appear out the integral sign as well. These examples can be classified as the **Volterra** and **Fredholm** i***ntegro-differential equations***. Equations (5.1) and (5.2) are the *Volterra* type whereas equations (5.3) and (5.4) are of *Fredholm* type integro-differential equations. It is to be noted that these equations are linear integro-differential equations. However, nonlinear integro-differential equations also arise in many scientific and engineering problems. Our concern in this chapter will be linear integro-differential equations and we will be concerned with the different solution techniques. To obtain a solution of the integro-differential equation, we need to specify the initial conditions to determine the unknown constants.

Note that, the above given equations involve the derivatives and the integral operators in the same equation, and consequently the term *integro-differential* equations has been used for problems involving this combination of operators. Moreover, because these equations involve differential operator, it is necessary to describe the initial conditions for each

problem. The initial conditions in solving any differential equation are needed to determine the constants of integration.

5.2 Volterra Integro-Differential Equations

In this section, we shall present some sophisticated mathematical methods that will be used to handle Volterra Integro-Differential Equations, given in the form

$$u^{(n)}(x) = f(x) + \lambda \int_0^x K(x,t)u(t)\,dt. \qquad (5.5)$$

We shall focus our attention to study the integral equation that involves **separable** kernel of the form

$$K(x,t) = \sum_{k=1}^{n} g_k\,(x)h_k(t)\,. \qquad (5.6)$$

Without loss of generality, we will consider the cases where the kernel $K(x,t)$ consists of one product of the functions $g(x)$ and $h(t)$ given by

$$K(x,t) = g(x)h(t)\,, \qquad (5.7)$$

where other cases can be generalized in the same manner. The **nonseparable** kernel can be reduced to separable kernel by using the Taylor expansion for the kernel involved. The methods to be introduced are identical, with some exceptions, to the methods discussed in **Chapter 3**. Our approach will be mainly based on how we can extend the methods used in Chapter 3 to handle this type of equations. For this reason we first start with the most practical method.

► 5.2.1 The Series Solution Method

This method has been extensively introduced in **Chapter 3**. Without loss of generality, we may consider a standard form to the Volterra integro-differential equation given by

$$u^{(n)}(x) = f(x) + \int_0^x K(x,t)u(t)\,dt, \qquad u^{(k)}(0) = b_k\,, 0 \le k \le (n-1) \quad (5.8)$$

where $u^{(n)}(x)$ indicates the *n*th derivative of $u(x)$ with respect to x, and b_k are constants that define the initial conditions. Substituting Eq. (5.7) into Eq. (5.8) yields

$$u^{(n)}(x) = f(x) + g(x)\int_0^x h(t)\,u(t)dt, \qquad u^{(k)}(0) = b_k\,, 0 \le k \le (n-1) \quad (5.9)$$

We will follow a manner parallel to the approach of the *series solution method* that usually used in solving ordinary differential equations around an ordinary point. To achieve this goal, we first assume that the solution $u(x)$ of Eq. (5.9) is an analytic function and hence can be represented by a series expansion given by

$$u(x) = \sum_{k=0}^{\infty} a_k\, x^k\,, \quad (5.10)$$

where the coefficients a_k are constants that will be determined. It is to be noted that the first few coefficients a_k can be determined by using the initial conditions so that

$$\left.\begin{aligned} a_0 &= u(0) \\ a_1 &= u'(0) \\ a_2 &= \frac{1}{2!}u''(0) \end{aligned}\right\} \quad (5.11)$$

and so on depending on the number of the initial conditions given, whereas the remaining coefficients a_k will be determined from applying the technique as will be discussed later. Substituting Eq. (5.10) into both sides of Eq. (5.9) yields

$$\left(\sum_{k=0}^{\infty} a_k\, x^k\right)^{(n)} = f(x) + g(x)\int_0^x h(t)\left(\sum_{k=0}^{\infty} a_k\, x^k\right)dt\,. \quad (5.12)$$

In view of Eq. (5.12), the integral equation Eq. (5.9) will be reduced to several calculable integrals in the right hand side of Eq. (5.12) that can be easily evaluated where we have to integrate terms of the form $t^n, n \ge 0$ only.

The next step is to write the Taylor expansion for $f(x)$, evaluate the resulting traditional integrals in Eq. (5.12), and then equating the coefficients of like powers of x in both sides of the equation. Accordingly, this leads to a complete determination of the coefficients $a_i, i \geq 0$

Consequently, substituting the obtained coefficients $a_k, k \geq 0$ into Eq. (5.10) produces the solution in a series form. This may converge to a solution in a closed form, if an exact solution exists, or we may use the obtained series for numerical purposes.

To give a clear overview of the technique and how it should be implemented for Volterra integro-differential equations, the series solution method will be illustrated by discussing the following example.

Example 1: Solve the following Volterra integro-differential equation

$$u''(x) = x\cosh(x) - \int_0^x tu(t)\,dt, \qquad u(0) = 0, u'(0) = 1. \tag{5.13}$$

by using the series solution method.

Solution: Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n\, x^n\,, \tag{5.14}$$

into both sides of the equation Eq.(5.13) and using the Taylor expansion of $\cosh(x)$ leads to

$$\sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-2} = x\left(\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}\right) - \int_0^x t\left(\sum_{n=0}^{\infty} a_n t^n\right) dt. \tag{5.15}$$

Using the initial conditions yields

$$a_0 = 0\,, \tag{5.16}$$

$$a_1 = 1\,. \tag{5.17}$$

Evaluating the traditional integrals that involve terms of the form $t^n, n \geq 0$, and using few terms from both sides yield

$$2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots = x\left(1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right)$$

$$-\left(1 + \frac{1}{3}x^3 + \frac{1}{4}a_2x^4 + \cdots\right) \qquad (5.18)$$

Equating the coefficients of like powers of *x* in both sides we find

$$\left.\begin{aligned} a_2 &= 0 \\ a_3 &= \frac{1}{3!} \\ a_4 &= 0 \end{aligned}\right\} \qquad (5.19)$$

and generally

$$a_{2n} = 0, \quad for\ n \geq 0 \qquad (5.20)$$

$$a_{2n+1} = \frac{1}{(2n+1)!} \quad for\ n \geq 0\,. \qquad (5.21)$$

Using Eq. (5.14) we find the solution $u(x)$ in a series form

$$u(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots, \qquad (5.22)$$

and in a closed form by

$$u(x) = sinh(x). \qquad (5.23)$$

obtained upon using the Taylor series of $sinh(x)$. ■

► 5.2.2 The Adomian Decomposition Method

The Adomian decomposition method (ADM) and the modified decomposition method were discussed in detail in **Chapters 3**. In this section we will introduce how this successful method can be implemented to determine a series solution to the Volterra integro differential equations. Without loss of generality, we may assume a standard form to the Volterra integro-differential equation defined by the standard form

$$u^{(n)}(x) = f(x) + \int_0^x K(x,t)u(t)\,dt, \qquad u^{(k)}(0) = b_k\,, 0 \le k \le (n-1) \quad (5.24)$$

where $u^{(n)}(x)$ indicates the *n*th derivative of $u(x)$ with respect to x, and b_k are constants that define the initial conditions. It is natural to seek an expression for $u(x)$ that will be derived from Eq. (5.24). This can be done by integrating both sides of Eq. (5.24) from 0 to x as many times as the order of the derivative involved. Consequently, we obtain

$$u(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}\big(f(x)\big) + L^{-1}\left(\int_0^x K(x,t)\,u(t)dt\right), \qquad (5.25)$$

where $\sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k$ is obtained by using the initial conditions, and L^{-1} is an *n-fold integration* operator. Now we apply the decomposition method by defining the solution $u(x)$ of Eq. (5.25) in a decomposition series given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (5.26)$$

Substituting Eq. (5.26) into both sides of Eq. (5.25) we get

$$\sum_{n=0}^{\infty} u_n(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}\big(f(x)\big) + L^{-1}\left(\int_0^x K(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt\right), \quad (5.27)$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}\big(f(x)\big)$$

$$+L^{-1}\left(\int_0^x K(x,t)\left(\sum_{n=0}^{\infty} u_0(t)\right)dt\right)$$

$$+L^{-1}\left(\int_0^x K(x,t)\left(\sum_{n=0}^{\infty} u_1(t)\right)dt\right)$$

$$+L^{-1}\left(\int_0^x K(x,t)\left(\sum_{n=0}^{\infty} u_2(t)\right)dt\right)$$

$$+\cdots \tag{5.28}$$

The components $u_i(x)\,, i \geq 0$ of the unknown function $u(x)$ are determined in a recursive manner, in a similar way as discussed before, if we set

$$u_0(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}\big(f(x)\big), \tag{5.29}$$

$$u_1(x) = L^{-1}\left(\int_0^x K(x,t)\left(\sum_{n=0}^{\infty} u_0(t)\right)dt\right), \tag{5.30}$$

$$u_2(x) = L^{-1}\left(\int_0^x K(x,t)\left(\sum_{n=0}^{\infty} u_1(t)\right)dt\right), \tag{5.31}$$

$$u_3(x) = L^{-1}\left(\int_0^x K(x,t)\left(\sum_{n=0}^{\infty} u_2(t)\right)dt\right), \tag{5.32}$$

and so on. The decomposition method discussed above for the determination of the components $u_i(x), i \geq 0$ of the solution $u(x)$ of the equation (5.24) can be written in a recursive manner by

$$u_0(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}\big(f(x)\big), \tag{5.33}$$

$$u_{n+1}(x) = L^{-1}\left(\int_0^x K(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt\right), \qquad n \geq 0. \tag{5.34}$$

In view of Eq. (5.33) and Eq. (5.34), the components $u_i(x), i \geq 0$ are immediately evaluated. With the components determined, the solution $u(x)$ of Eq. (5.24) is then obtained in a series form using Eq. (5.26). Consequently, the series obtained for $u(x)$ mostly provides the exact solution in a closed form as will be illustrated later. However, for concrete problems, where Eq. (5.26) cannot be evaluated, a truncated series is usually used to approximate the solution $u(x)$.

It is convenient to point out that the phenomenon of the self-cancelling noise terms that was introduced before may be applied here if the noise terms appear between $u_0(x)$ and $u_1(x)$. Moreover, the modified decomposition method can be used as well. The following example will explain how we can use the Adomian decomposition technique.

Example 1: Solve the following Volterra integro-differential equation by using ADM.

$$u''(x) = x + \int_0^x (x-t)u(t)\,dt, \qquad u(0) = 0, u'(0) = 1. \tag{5.35}$$

Solution: Applying the *two-fold* integral operator L^{-1}

$$L^{-1}(\cdot) = \int_0^x \int_0^x (\cdot)\,dxdx \tag{5.36}$$

to both sides of Eq. (5.35), i.e. integrating both sides of Eq. (5.35) twice from 0 to x, and using the given initial conditions yield

$$u(x) = x + \frac{1}{3!}x^3 + L^{-1}\left(\int_0^x (x-t)u(t)\,dt\right). \tag{5.37}$$

Following the decomposition scheme Eq. (5.33) and Eq. (5.34) we find

$$u_0(x) = x + \frac{1}{3!}x^3 \ , \tag{5.38}$$

$$u_1(x) = L^{-1}\left(\int_0^x (x-t)u_0(t)\,dt\right) = \frac{1}{5!}x^5 + \frac{1}{7!}x^7, \tag{5.39}$$

$$u_2(x) = L^{-1}\left(\int_0^x (x-t)u_1(t)\,dt\right) = \frac{1}{9!}x^9 + \frac{1}{11!}x^{11}, \tag{5.40}$$

$$u_3(x) = L^{-1}\left(\int_0^x (x-t)u_2(t)\,dt\right) = \frac{1}{13!}x^{13} + \frac{1}{15!}x^{15}\ , \tag{5.41}$$

and so on. Combining the last results yields the solution $u(x)$ in a series form given by

$$u(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \frac{1}{11!}x^{11} + \frac{1}{13!}x^{13} + \frac{1}{15!}x^{15} + \cdots \tag{5.42}$$

and this leads to the exact solution in a closed form

$$u(x) = sinh(x). \blacksquare \tag{5.43}$$

Remark 5.1: By using the Laplace transform method with the concept of convolution and using the initial conditions the given equation can be very easily simplified to

$$\mathcal{L}\{u(x)\} = \frac{1}{s^2 - 1} \tag{5.44}$$

and taking the inverse transform, we obtain $u(x) = sinh(x)$ which is identical to the previous result.

► 5.2.3 The Variational Iteration Method

The variational iteration method was used effectively in the preceding chapters. This method was used to handle both Fredholm and Volterra integral equations. The method was presented before where we studied the necessary issues that should be addressed. We should first give the correction functional that works for differential equation provided that the Lagrange multiplier is derived. The method gives successive approximations of the exact solution and does not give distinct components as in the case of Adomian decomposition method. Because we will study the Volterra integro-differential equations in this section, then we can apply this method directly, and we do not need any specific condition. In what follows we give a brief summary to the essential steps of the variational method.

The standard *j*th order integro-differential equation is of the form

$$u^{(j)}(x) = f(x) + \int_0^x K(x,t)u(t)\,dt, \qquad (5.45)$$

where $u^{(j)}(x) = \frac{d^j u}{dx^j}$ and $u(0), u'(0), u''(0), \dots, u^{j-1}(0)$ are the initial conditions.

The correction functional for the integro-differential equation Eq. (5.45) is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(x,t)\left(u_n^{(i)}(t) - f(t) - \int_0^t K(t,r)u_n(r)\,dr \right) dt \qquad (5.46)$$

Recall that the Lagrange multipliers were identified optimally, as shown before, and summarized by

$$u' + f\big(u(\xi), u'(\xi)\big) = 0, \qquad \lambda = -1\,, \qquad (5.47)$$

$$u'' + f\big(u(\xi), u'(\xi), u''(\xi)\big) = 0, \qquad \lambda = \xi - x\,, \qquad (5.48)$$

$$u''' + f\big(u(\xi), u'(\xi), u''(\xi), u'''(\xi)\big) = 0, \qquad \lambda = -\frac{1}{2!}(\xi - x)^2\,, \qquad (5.49)$$

and so on (see **Chapter 3**, sec. 3.7). Moreover, the zeroth approximation $u_0(x)$ can be preferably selected by using the initial values $u(0), u'(0), u''(0), \ldots$ and Taylor series. Having determined the first few successive approximations, the solution is given by

$$u(x) = \lim_{n\to\infty} u_n(x) \, . \tag{5.50}$$

The VIM will be illustrated by studying the following example.

Example 1: Solve the Volterra integro-differential equation by using the VIM.

$$u'(x) = -x + \frac{1}{2}x^2 \; + \int_0^x u(t)\, dt, \qquad u(0) = 2 \, . \tag{5.51}$$

Solution: The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) + t - \frac{1}{2}t^2 - \int_0^t u_n(r)\, dr \right) dt \tag{5.52}$$

where we used $\lambda = -1$ for first-order integro-differential equations.

We can use the initial condition to select $u_0(x) = u(0) = 2$. Using this selection into the correction functional gives the following successive approximations

$$u_0(x) = 2 \, , \tag{5.53}$$

$$u_1(x) = u_0(x) - \int_0^x \left(u_0'(t) + t - \frac{1}{2}t^2 - \int_0^t u_0(r)\, dr \right) dt$$

$$= 2 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \, , \tag{5.54}$$

$$u_2(x) = u_1(x) - \int_0^x \left(u_1'(t) + t - \frac{1}{2}t^2 - \int_0^t u_1(r)\, dr \right) dt$$

$$= 2 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 , \qquad (5.55)$$

$$u_3(x) = u_2(x) - \int_0^x \left(u_2'(t) + t - \frac{1}{2}t^2 - \int_0^t u_2(r)\, dr \right) dt$$

$$= 2 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 , \qquad (5.56)$$

and so on. The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x) . \qquad (5.57)$$

that gives the exact solution

$$u(x) = 1 - x + e^x . \blacksquare \qquad (5.58)$$

Example 2: Solve the Volterra integro-differential equation by using the VIM.

$$u''(x) = -x + \int_0^x (x - t)u(t)\, dt, \qquad u(0) = 0 , \qquad u'(0) = 1. \qquad (5.59)$$

Solution: The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x (t - x) \left(u_n''(t) + t - \int_0^t (t - r)u_n(r)\, dr \right) dt \qquad (5.60)$$

where we used $\lambda = (t - x)$ for second-order integro-differential equations. We can use the initial condition to select $u_0(x) = u(0) + xu'(0) = x$. Using this selection into the correction functional gives the following successive approximations

$$u_0(x) = x , \qquad (5.61)$$

$$u_1(x) = u_0(x) - \int_0^x (t-x)\left(u_0''(t) + t - \int_0^t (t-r)u_0(r)\,dr \right) dt$$

$$= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5\,, \tag{5.62}$$

$$u_2(x) = u_1(x) - \int_0^x (t-x)\left(u_1''(t) + t - \int_0^t (t-r)u_1(r)\,dr \right) dt$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9\,,, \tag{5.63}$$

and so on. The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x)\,. \tag{5.64}$$

that gives the exact solution

$$u(x) = sin(x)\,. \blacksquare \tag{5.65}$$

► 5.2.4 Converting to Volterra Integral Equation

This section is concerned with converting to Volterra integral equations. We can easily convert the Volterra integro-differential equation to equivalent Volterra integral equation, **provided** the kernel is a difference kernel defined by $K(x,t) = K(x-t)$. This can be easily done by integrating both sides of the equation and using the initial conditions. To perform the conversion to a standard Volterra integral equation, we should use the following well-known formula Eq. (2.45) described in **Chapter 2** that converts multiple integrals into a single integral. We illustrate for the benefit three specific formulas:

$$\int_0^x \int_0^x u(t)\,dtdt = \int_0^x (x-t)\,u(t)dt\,, \tag{5.66}$$

$$\int_0^x \int_0^x u(t)\,dtdt = \int_0^x (x-t)\,u(t)dt\,, \tag{5.66}$$

$$\int_0^x \int_0^x \int_0^x u(t)\,dtdtdt = \frac{1}{2!}\int_0^x (x-t)^2\,u(t)dt\,, \tag{5.67}$$

$$\underbrace{\int_0^x \int_0^x \ldots \int_0^x u(t)\,dt}_{n\ fold\ integration} = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}\,u(t)dt \tag{5.68}$$

Having established the transformation to a standard Volterra integral equation, we may proceed using any of the alternative methods that were discussed before in **Chapter 3**. To give a clear overview of this method we discuss the following examples.

Example 1: Solve the following Volterra integro-differential equation by converting it to a standard Volterra integral equation.

$$u'(x) = 2 - \frac{1}{4}x^2 + \frac{1}{4}\int_0^x u(t)\,dt, \qquad u(0) = 0\,. \tag{5.69}$$

Solution: Integrating both sides from 0 to x and using the initial condition we obtain

$$u(x) = 2x - \frac{1}{12}x^3 + \frac{1}{4}\int_0^x \int_0^x u(t)\,dt \tag{5.70}$$

which gives

$$u(x) = 2x - \frac{1}{12}x^3 + \frac{1}{4}\int_0^x (x-t)\,u(t)dt \tag{5.71}$$

upon using the formula Eq. (5.66). It is clearly seen that Eq. (5.71) is a standard Volterra integral equation that will be solved by using the decomposition method. Following the Adomian decomposition method we set

$$u_0(x) = 2x - \frac{1}{12}x^3 \tag{5.72}$$

$$u_1(x) = \frac{1}{4}\int_0^x (x-t)\left(2t - \frac{1}{12}t^3\right)dt$$

$$= \frac{1}{12}x^3 - \frac{1}{240}x^5, \tag{5.73}$$

We can easily observe that the noise terms $\mp\frac{1}{12}x^3$ appear in the components $u_0(x)$ and $u_1(x)$, and by cancelling this noise term from $u_0(x)$ and justifying that

$$u(x) = 2x\,, \tag{5.74}$$

is the exact solution of Eq. (5.71) . ■

Example 2: Solve the following Volterra integro-differential equation by converting it to a standard Volterra integral equation.

$$u''(x) = -1 + \int_0^x (x-t)u(t)\,dt, \qquad u(0) = 1\,, \qquad u'(0) = 0\,. \tag{5.75}$$

Solution: Integrating both sides from 0 to x and using the initial condition we obtain

$$u'(x) = -x + \int_0^x \int_0^x (x-t)u(t)\,dt \tag{5.76}$$

$$= -x + \frac{1}{2!}\int_0^x (x-t)^2 u(t)\,dt \tag{5.77}$$

by using formula Eq. (5.67), and

$$u(x) = 1 - \frac{1}{2!}x^2 + \int_0^x \int_0^x \int_0^x (x-t)\,u(t)dtdtdt \tag{5.78}$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{3!}\int_0^x (x-t)^3\, u(t)dt, \qquad (5.79)$$

by using formula Eq. (5.67).
The last equation is a standard Volterra integral equation that will be solved by using the modified decomposition method. To determine $u(x)$ we set

$$u_0(x) = 1\,, \qquad (5.80)$$

which gives

$$u_1(x) = -\frac{1}{2!}x^2 + \frac{1}{4!}x^4, \qquad (5.81)$$

We can easily observe that the noise terms **did not appear** between the components $u_0(x)$ and $u_1(x)$, therefore we continue to find more terms to study the solution more closely. Consequently, we find

$$u_2(x) = -\frac{1}{6!}x^6 + \frac{1}{8!}x^8, \qquad (5.82)$$

Combining the results for the components $u_0(x)$, $u_1(x)$ and $u_2(x)$ we obtain the series expression for the solution given by

$$u(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots \qquad (5.83)$$

which gives the exact solution

$$u(x) = cos(x)\,. \blacksquare \qquad (5.84)$$

► 5.2.5 Converting to Initial Value Problems

In this section we will study how to convert the Volterra integro-differential equation to an equivalent initial value problem, focusing our discussion on the case where the kernel is a difference kernel where $K(x,t) = K(x-t)$. This can be achieved easily by differentiating both sides of the integro-differential equation as many times as needed to remove the integral

sign. In differentiating the integral involved we shall use the Leibniz rule to achieve our goal. The Leibniz rule has been extensively introduced in **Chapter 2**, sec. 5.1. It is important to note that we should define the initial conditions at every step of differentiation. A similar technique was discussed and examined (**see Chapter 2**).

Having converted the Volterra integro-differential equation to an initial value problem, the various methods that are used in any ordinary differential equation course can be used to determine the solution. The idea is easy to use but requires more calculations if compared with the integral equations techniques.

To give a clear overview of this method we discuss the following illustrative examples.

Example 1: Solve the following Volterra integro-differential equation by converting it to an initial value problem.

$$u'(x) = 1 + \int_0^x u(t)\,dt, \qquad u(0) = 0\,. \tag{5.85}$$

Solution: Differentiating both sides of Eq. (5.85) with respect to x and using the Leibniz rule to differentiate the integral at the right hand side we obtain

$$u''(x) = u(x)\,, \tag{5.86}$$

with initial conditions given by

$$u(0) = 0\,, \qquad u'(0) = -1\,, \tag{5.87}$$

where the last initial condition was obtained by substituting $x = 0$ in both sides of Eq. (5.85). The characteristic equation of Eq. (5.85) is

$$r^2 - 1 = 0\,, \tag{5.88}$$

which gives the roots

$$r = \mp 1\,. \tag{5.89}$$

So, the general solution is given by

$$u(x) = Acosh(x) + Bsinh(x) \tag{5.90}$$

where *A* and *B* are constants to be determined. Using the initial conditions given by Eq.(5.87) to find the constants *A* and *B*, we find that the solution is

$$u(x) = sinh(x)\,. \blacksquare \tag{5.91}$$

Example 2: Solve the following Volterra integro-differential equation by converting it to an initial value problem.

$$u''(x) = -x + \int_0^x (x-t)u(t)\,dt, \qquad u(0) = 0\,, \qquad u'(0) = 1\,. \tag{5.92}$$

Solution: Differentiating both sides of Eq. (5.92) with respect to x and using the Leibniz rule to differentiate the integral at the right hand side we obtain

$$u'''(x) = -1 + \int_0^x u(t)\,dt\,, \tag{5.93}$$

and by differentiating again to reduce the equation to a pure differential equation we obtain

$$u^{(iv)}(x) = u(x)\,. \tag{5.94}$$

Combining the given initial conditions in Eq. (5.92) with the other initial conditions, obtained by substituting $x = 0$ in Eq. (5.92) and Eq. (5.93), we write

$$u(0) = 0\,, u'(0) = 1, u''(0) = 0\,, u'''(0) = -1\,. \tag{5.95}$$

The characteristic equation of Eq. (5.94) is

$$r^4 - 1 = 0\,, \tag{5.96}$$

which gives the roots

$$r = \mp 1\,, \mp i\,. \tag{5.97}$$

So, the general solution is given by

$$u(x) = Acosh(x) + Bsinh(x) + Ccos(x) + Dsin(x)\,, \qquad (5.98)$$

where *A, B, C, and D* are constants to be determined. Using the initial conditions given by Eq.(5.95) to find the constants *A, B, C,* and *D*, we find that the solution is

$$u(x) = sin(x)\,. \blacksquare \qquad (5.99)$$

Example 3: Solve the following Volterra integro-differential equation by reducing the equation to an initial value problem.

$$u'(x) = 2 - \frac{1}{4}x^2 + \frac{1}{4}\int_0^x u(t)\,dt, \qquad u(0) = 0\,. \qquad (5.100)$$

Solution: Differentiating both sides of Eq. (5.100) with respect to x and using the Leibniz rule to differentiate the integral at the right hand side we obtain

$$u''(x) = \frac{-1}{2}x + \frac{1}{4}u(x)\,, \qquad (5.101)$$

or equivalently

$$u''(x) - \frac{1}{4}u(x) = \frac{-1}{2}x\,, \qquad (5.102)$$

with initial conditions

$$u(0) = 0\,, \qquad u'(0) = 2\,, \qquad (5.103)$$

where the second initial condition was obtained by substituting $x = 0$ in both sides of Eq. (5.100). The characteristic equation for the corresponding homogeneous equation of Eq. (5.102) is

$$r^2 - \frac{1}{4} = 0\,, \qquad (5.104)$$

which gives the roots

$$r = \mp\frac{1}{2}\,. \qquad (5.105)$$

so that the complementary solution is given by

$$u_c(x) = A\cos\left(\frac{x}{2}\right) + B\sin\left(\frac{x}{2}\right), \qquad (5.106)$$

where A and B are constants to be determined. A particular solution $u_p(x)$ can be obtained by assuming that

$$u_p(x) = C + Dx\,. \qquad (5.107)$$

Substituting Eq. (5.107) into Eq. (5.102) yields

$$C = 0\,, \qquad D = 2\,. \qquad (5.108)$$

Combining Eqs. (5.106) - (5.108) yields

$$u(x) = A\cos\left(\frac{x}{2}\right) + B\sin\left(\frac{x}{2}\right) + 2x\,, \qquad (5.109)$$

Which gives

$$u(x) = 2x\,. \qquad (5.110)$$

Upon using the initial conditions Eq. (5.103). ■

► 5.2.6 Volterra Integro-Differential Equations of the First Kind

The standard form of the Volterra integro-differential equation of the first kind was introduced by Linz **[25, 26]**. The standard form of this integro-differential equation reads

$$\int_0^x K_1(x,t)\,u(t)dt + \int_0^x K_2(x,t)\,u^{(n)}(t)dt = f(x), \qquad K_2(x,t) \neq 0\,, \qquad (5.111)$$

with given initial conditions. The Volterra integro-differential equation of the first kind Eq. (5.111) was investigated by Linz **[3,4]** by using analytical and numerical methods. Moreover, the equation was handled by other techniques as will be presented later.

The most significant method for handling this problem is to convert it to an equivalent Volterra integro-differential equation of the second kind, or even Volterra integral equation

of the second kind simply by using the Leibniz rule for differentiating any integral. Also this can be achieved by integrating the second integral in Eq. (5.111) by parts.
Having converted the Volterra integro-differential equation of the first kind to its equivalent of the second kind, then we can employ any of the methods that were applied earlier in previous chapters, such as the series method, the Adomian decomposition method and the variational iteration method. For practical use, we will apply the latter method. Also, you are advised to use other methods.

In this Chapter we will concern ourselves on the Volterra integro-differential equation of the first kind where the kernels $K_1(x,t)$ and $K_2(x,t)$ of Eq. (5.111) are *difference kernels*, that each depends on the difference $(x-t)$ such as $(x-t+1)$, $sin(x-t)$, e^{x-t} ... .

One more essential condition that should be considered that the solution exists if $K_2(x,t) \neq 0$. This necessary condition can be observed by introducing the conversion process to an equivalent Volterra integro-differential equation of the second kind.

Differentiating both sides of Eq. (5.111), and using Leibniz rule we find

$$u^{(n)}(x) = \frac{f'(x)}{K_2(x,x)} - \frac{K_1(x,x)}{K_2(x,x)}u(x) - \frac{1}{K_2(x,x)}\int_0^x \frac{\partial\big(K_1(x,t)\big)}{\partial x}u^{(n)}(t)\,dt$$

$$-\frac{1}{K_2(x,x)}\int_0^x \frac{\partial\big(K_2(x,t)\big)}{\partial x}u^{(n)}(t)\,dt, \qquad (5.112)$$

The obtained equation is a Volterra integro-differential equation of the second kind that can be evaluated only if $K_2(x,x) \neq 0$.

We will focus our illustrative examples by the variational iteration method, where other methods can be used as well. Recall that we need the Lagrange multiplier λ that will be used in the correction functional. Moreover, the zeroth approximation can be selected by using the given initial condition as presented before. The Volterra integro-differential equation of the first kind will be examined by using the variational iteration method when studying the following examples.

Example 1: Solve the Volterra integro-differential equation of the first kind

$$\int_0^x (x-t+1)\,u'(t)dt = 2e^x - x - 2\,,\;\; u(0) = 1\,. \tag{5.113}$$

Solution: Differentiating both sides of this equation once with respect to x gives the Volterra integro-differential equation of the second kind

$$u'(x) = 2e^x - 1 - \int_0^x u'(t)\,dt, \tag{5.114}$$

The correction functional for Eq. (5.114) is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - 2e^t + 1 + \int_0^t u_n'(r)\,dr \right) dt \tag{5.115}$$

where we used $\lambda(t) = -1$. The zeroth approximation $u_0(x)$ can be selected by $u_0(x) = 1$. This gives the successive approximations

$$\left.\begin{aligned} u_0(x) &= 1, \\ u_1(x) &= 2e^x - 1 - x, \\ u_2(x) &= 1 + x + \frac{1}{2!}x^2, \\ u_3(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4, \\ &\vdots \\ &\vdots \end{aligned}\right\} \tag{5.116}$$

This gives

$$u_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \tag{5.117}$$

that converges to the exact solution

$$u(x) = e^x\,. \;\blacksquare \tag{5.118}$$

Example 2: Solve the Volterra integro-differential equation of the first kind

$$\int_0^x (x-t)u(t)dt + \int_0^x (x-t+1)\,u'(t)dt = 1 + x - cos(x)\,,\;\; u(0) = 0\,. \qquad (5.119)$$

Solution: Differentiating both sides of this equation once with respect to x gives the Volterra integro-differential equation of the second kind

$$u'(x) = 1 + sin(x) - \int_0^x \big(u(t) + u'(t)\big)\,dt. \qquad (5.120)$$

The correction functional for Eq. (5.120) is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - 1 - sin(t) + \int_0^t \big(u_n(r) + u_n'(r)\big)\,dr \right) dt. \qquad (5.121)$$

The zeroth approximation $u_0(x)$ can be selected by $u_0(x) = 0$. This gives the successive approximations

$$\left.\begin{aligned} u_0(x) &= 0, \\ u_1(x) &= 1 + 1 - cos(x), \\ u_2(x) &= x - \frac{1}{3!}x^3 - \frac{1}{12!}x^4 + \frac{1}{5!}x^5 + \cdots, \\ u_3(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots, \\ &\vdots \\ &\vdots \end{aligned}\right\} \qquad (5.122)$$

This gives the exact solution

$$u(x) = sin(x)\,. \;\blacksquare \qquad (5.123)$$

5.3 Fredholm Integro-Differential Equations

In this section, we will discuss the reliable methods used to solve Fredholm integro-differential equations, where both differential and integral operators will appear in the same equation. The differential operator may be of first order or higher order. This type of equations was termed as Fredholm integro-differential equations, given in the form

$$u^{(n)}(x) = f(x) + \lambda \int_a^b K(x,t)u(t)\,dt\,. \qquad (5.124)$$

We remark here that we will focus our attention on the equations that involve separable kernels where the kernel $K(x,t)$ can be expressed as the finite sum of the form

$$K(x,t) = \sum_{k=1}^{n} g_k\,(x)h_k(t)\,. \qquad (5.125)$$

Without loss of generality, we will consider the cases where the kernel $K(x,t)$ consists of one product of the functions $g(x)$ and $h(t)$ given by

$$K(x,t) = g(x)h(t)\,, \qquad (5.126)$$

where other cases can be generalized in the same manner. The **nonseparable** kernel can be reduced to separable kernel by using the Taylor expansion for the kernel involved.

The Fredholm integro-differential equations are usually solved by a variety of methods, some are numerical methods, whereas others are analytic methods. It is worth noting that in this section we will introduce the most recent and practical schemes that handle this type of equations, where we may obtain an exact solution or an approximation to the solution with the highest desirable accuracy.
The methods to be introduced are identical, with some exceptions, to the methods discussed in **Chapter 4**. Our approach will be mainly based on how we can extend the methods used in Chapter 4 to handle this type of equations. For this reason we first start with the most practical method.

We point out that the direct computation method, that will be used first, requires integrating or multiple integrating of the derivatives that are involved in the equation. In what follows, we list some of the facts used in calculus courses:

$$\int_0^x u'(t)dt = u(x) - u(0), \tag{5.127}$$

$$\int_0^x \int_0^t u''(s)\, dsdt = u'(x) - u(0) - xu'(0), \tag{5.128}$$

$$\int_0^x \int_0^t \int_0^s u'''(r)\, drdsdt = u'(x) - u(0) - xu'(0) - \frac{1}{2}x^2u''(0), \tag{5.129}$$

and so on for other derivatives.

► 5.3.1 The Direct Computation Method

The direct computation method has been extensively used in **Chapter 4**, sec.4.5; to handle Fredholm integral equations. Without loss of generality, we may assume a standard form to the Fredholm integro-differential equation given by

$$u^{(n)}(x) = f(x) + \lambda \int_0^1 K(x,t)u(t)\, dt\,, \qquad u^{(k)}(0) = b_k, 0 \le k \le (n-1). \tag{5.130}$$

where $u^{(n)}(x)$ indicates the *n*th derivative of $u(x)$ with respect to x, and b_k are constants that define the initial conditions. Substituting Eq. (5.126) into Eq. (5.130) yields

$$u^{(n)}(x) = f(x) + g(x)\int_0^1 h(t)\,u(t)dt, \qquad u^{(k)}(0) = b_k\,, 0 \le k \le (n-1). \quad (5.131)$$

We can easily observe that the definite integral in the integro-differential equation (5.131) involves an integrand that completely depends on the variable t, and therefore, it seems reasonable to set that definite integral in the right side of Eq. (5.131) to a constant α, i.e. we set

$$\alpha = \int_0^1 h(t)\,u(t)dt\,. \quad (5.132)$$

With α defined in Eq. (5.132), the Eq. (5.131) can be written by

$$u^{(n)}(x) = f(x) + \alpha\, g(x)\,. \quad (5.133)$$

It remains to determine the constant α to evaluate the exact solution $u(x)$. To find α, we should derive a form for $u(x)$ by using Eq.(5.133), followed by substituting this form in Eq. (5.132). To achieve this we integrate both sides of Eq. (5.133) *n* times from 0 to x, and by using the given initial conditions $u^{(k)}(0) = b_k\,, 0 \le k \le (n-1)$ we obtain an expression for $u(x)$ given by

$$u(x) = p(x;\alpha)\,, \quad (5.134)$$

where $p(x;\alpha)$ is the result derived from integrating Eq. (5.133) and by using the given initial conditions. Substituting Eq. (5.134) into the right hand side of Eq. (5.132), integrating and solving the resulting equation lead to a complete determination of α. The exact solution of Eq. (5.130) follows immediately upon substituting the resulting value of α into Eq. (5.134).

To give a clear view of the technique, we illustrate the method by solving the following examples.

Example 1: Solve the first-order Fredholm integro-differential equation by using the direct computation method.

$$u'(x) = 1 - \frac{1}{3}x + x\int_0^1 tu(t)\,dt\,, \qquad u(0) = 0. \tag{5.135}$$

Solution: The equation (5.135) may be written in the form

$$u'(x) = 1 - \frac{1}{3}x + \alpha x\,, \qquad u(0) = 0\,, \tag{5.136}$$

where the constant α is defined by

$$\alpha = \int_0^1 tu(t)\,dt\,. \tag{5.137}$$

To determine α, we first need an expression for $u(x)$ to be used in Eq. (5.137). This can be easily done by integrating both sides of Eq. (5.136) from 0 to x, using Eq. (5.127), and by using the given initial condition we obtain

$$u(x) = x + \left(\frac{\alpha}{2} - \frac{1}{6}\right)x^2\,. \tag{5.138}$$

Substituting Eq. (5.138) into Eq. (5.137) and evaluating the integral we find

$$\alpha = \frac{1}{3}\,, \tag{5.139}$$

so that the exact solution

$$u(x) = x\,, \tag{5.140}$$

follows immediately upon using Eq. (5.139) into Eq. (5.138). ■

Example 2: Solve the following third-order Fredholm integro-differential equation by using the direct computation method.

$$u'''(x) = sin(x) - x - \int_0^{\frac{\pi}{2}} xtu'(t)\,dt\,, u(0) = 1, u'(0) = 0, u''(0) = -1. \tag{5.141}$$

Solution: The equation (5.141) may be written in the form

$$u'''(x) = sin(x) - (1+\alpha)x\,, u(0) = 1, u'(0) = 0, u''(0) = -1. \qquad (5.142)$$

where the constant α is defined by

$$\alpha = \int_0^{\frac{\pi}{2}} tu'(t)\,dt\,. \qquad (5.143)$$

To determine α, we first need an expression for $u'(x)$ in terms of x and α to be used in Eq. (5.143). This can be easily done by integrating both sides of Eq. (5.142) three times from 0 to x, and using the given initial condition we obtain

$$u''(x) = -cos(x) - \frac{1+\alpha}{2!}x^2\,, \qquad (5.144)$$

$$u'(x) = -sin(x) - \frac{1+\alpha}{3!}x^3\,, \qquad (5.145)$$

and

$$u(x) = cos(x) - \frac{1+\alpha}{4!}x^4\,, \qquad (5.146)$$

Substituting Eq. (5.145) into Eq. (5.143) we obtain

$$\alpha = \int_0^{\frac{\pi}{2}} \left(-t\,sin(t) - \frac{1+\alpha}{3!}t^4\right)dt\,, \qquad (5.147)$$

which gives

$$\alpha = -1\,. \qquad (5.148)$$

Substituting Eq. (5.148) into Eq. (5.146) gives the exact solution

$$u(x) = cos(x)\,, \ \blacksquare \qquad (5.149)$$

► 5.3.2 The Adomain Decomposition Method

The Adomian decomposition method in its simplest form, has been extensively introduced in **Chapters 3 and 4** for handling Fredholm and Volterra integral equations. In this section we will study how this powerful method can be implemented to determine a series solution to the Fredholm integro-differential equations. As indicated earlier, we may assume a standard form to the Fredholm integro-differential equation given by

$$u^{(n)}(x) = f(x) + \lambda \int_0^1 K(x,t)u(t)\,dt\,, \qquad u^{(k)}(0) = b_k, 0 \le k \le (n-1). \quad (5.150)$$

where $u^{(n)}(x)$ indicates the *n*th derivative of $u(x)$ with respect to x, and b_k are constants that define the initial conditions. Substituting Eq. (5.126) into Eq. (5.150) yields

$$u^{(n)}(x) = f(x) + g(x) \int_0^1 h(t)\,u(t)dt, \qquad u^{(k)}(0) = b_k\,, 0 \le k \le (n-1). \quad (5.151)$$

We can easily observe that the definite integral in the integro-differential equation (5.151) involves an integrand that completely depends on the variable t as discussed in the preceding section. In an operator form, the equation (5.151) can be written as

$$Lu(x) = f(x) + g(x) \int_0^1 h(t)\,u(t)dt, \quad (5.152)$$

where the differential operator *L* is given by

$$L = \frac{d^n}{dx^n}\,. \quad (5.153)$$

It is clear that *L* is an invertible operator, therefore the integral operator L^{-1} is an *n-fold* integration operator and may be considered as definite integrals from 0 to x for each integral. Applying L^{-1} to both sides of Eq. (5.152) yields

$$u(x) = b_0 + b_1 x + \frac{1}{2!} b_2 x^2 + \cdots + \frac{1}{(n-1)!} b_{n-1} x^{(n-1)} + L^{-1}\big(f(x)\big)$$

$$+ \left(\int_0^1 h(t)\, u(t) dt \right) L^{-1}\big(g(x)\big)\,. \qquad (5.154)$$

In other words we integrated Eq. (5.152) n times from 0 to x and we used the initial conditions at every step of integration. It is important to note that the equation obtained in Eq. (5.154) is a standard Fredholm integral equation. This note will be used in the coming section. In the decomposition method we usually define the solution $u(x)$ of Eq.(5.150) in a series form given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x)\,. \qquad (5.155)$$

Substituting Eq. (5.155) into both sides of Eq. (5.154) we get

$$\sum_{n=0}^{\infty} u_n(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}\big(f(x)\big) + \left(\int_0^1 h(t) \left(\sum_{n=0}^{\infty} u_n(t) \right) dt \right) L^{-1}\big(g(x)\big). \quad (5.156)$$

or equivalently

$$\left.\begin{aligned} u_0(x) + u_1(x) + u_2(x) + \cdots = {} & \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}\big(f(x)\big) \\ & + \left(\int_0^1 h(t)\, u_0(t) dt \right) L^{-1}\big(g(x)\big) \\ & + \left(\int_0^1 h(t)\, u_1(t) dt \right) L^{-1}\big(g(x)\big) \\ & + \left(\int_0^1 h(t)\, u_2(t) dt \right) L^{-1}\big(g(x)\big) \\ & + \cdots \end{aligned}\right\} \quad (5.157)$$

The components $u_i(x)\,, i \geq 0$ of the unknown function $u(x)$ are determined in a recurrent manner, in a similar fashion as discussed before, where we set

$$u_0(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}\big(f(x)\big)\,, \tag{5.158}$$

$$u_1(x) = \left(\int_0^1 h(t)\, u_0(t) dt \right) L^{-1}\big(g(x)\big)\,, \tag{5.159}$$

$$u_2(x) = \left(\int_0^1 h(t)\, u_1(t) dt \right) L^{-1}\big(g(x)\big)\,, \tag{5.160}$$

$$u_3(x) = \left(\int_0^1 h(t)\, u_2(t) dt \right) L^{-1}\big(g(x)\big)\,, \tag{5.161}$$

and so on. The above discussed scheme for the determination of the components $u_0(x), u_1(x), u_2(x), u_3(x), \ldots$ of the solution $u(x)$ of the equation (5.150) can be written in a recursive relationship by

$$u_0(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}\big(f(x)\big)\,, \tag{5.162}$$

$$u_{n+1}(x) = \left(\int_0^1 h(t)\, u_n(t) dt \right) L^{-1}\big(g(x)\big)\,, \;\; n \geq 0\,. \tag{5.163}$$

In view of Eq. (5.163), the components $u_0(x), u_1(x), u_2(x), u_3(x), \ldots$ of $u(x)$ are immediately determined. With these components established, the solution $u(x)$ of Eq. (5.150) is readily determined in a series form using Eq. (5.155). Consequently, the series obtained for $u(x)$ frequently provides the exact solution in a closed form as will be illustrated later. However, for some problems, where a closed form is not easy to find, we use the series form obtained to approximate the solution. We point out here that few terms of the series derived by the decomposition method usually provide the higher accuracy level of the approximate solution.

The *decomposition method* avoids massive computational work and difficulties that arise from other methods. The computational work can be minimized, sometimes, by using the *modified decomposition method* or by observing the so-called *self-cancelling noise terms phenomena*. The aforementioned techniques were presented earlier in details, hence it will be summarized in the following sections.

► 5.3.2.1 The Modified Decomposition Method

The modified decomposition method is a powerful technique that minimizes the size of calculations. An essential requirement for the use of this method is that the data function $f(x)$ in Eq. (5.152) should consist of more than one term. Consequently, the data function $f(x)$ can be decomposed to two parts of the form

$$f(x) = f_0(x) + f_1(x)\,. \tag{5.164}$$

As stated before, the selection of these two parts depends mainly on a trial basis. Moreover, using this selection, we set the modified recurrence relation as

$$\left.\begin{aligned} u_0(x) &= \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}\big(f(x)\big)\,, \\ u_1(x) &= L^{-1}\big(f_1(x)\big) + \left(\int_0^1 h(t)\,u_0(t)dt\right) L^{-1}\big(g(x)\big)\,, \\ u_{n+1}(x) &= \left(\int_0^1 h(t)\,u_n(t)dt\right) L^{-1}\big(g(x)\big), \qquad n \geq 1. \end{aligned}\right\} \tag{5.165}$$

Although the change between the Adomian decomposition method is slight, but it was proved from using the proposed modification that this techniques minimizes the calculations size, and mostly exact solution can be derived in using two iterations only for $u_0(x)$ and $u_1(x)$. The strength of this modified method was confirmed before in **Chapters 3 and 4**.

Another observation that facilitates the convergence of the solution is the so-called noise terms phenomenon that was explained before, but will be summarized next.

► 5.3.2.2 The Noise Terms Phenomenon

The phenomena of the self-cancelling noise terms was introduced by Adomian and Rach **[5]** and it was proved that the exact solution of any integral or integro-differential equation, for some cases, may be obtained by considering the first two components $u_0(x)$ and $u_1(x)$ only. This scheme was discussed extensively in previous chapters and used effectively in the literature. Instead of evaluating several components, it is useful to examine the first two components $u_0(x)$ and $u_1(x)$. The conclusion made in **[6]** suggests that if we observe the appearance of like terms in both components with opposite signs, then by cancelling these terms, the remaining non-cancelled terms of $u_0(x)$ may in some cases provide the exact solution. This can be justified through substitution. The self-cancelling terms, the identical terms with opposite terms, between the components $u_0(x)$ and $u_1(x)$ are called **the noise terms**. It was formally proved that other terms in other components will vanish in the limit if the noise terms occurred in $u_0(x)$ and $u_1(x)$ However, if the exact solution was not attainable by using this phenomena, then we should continue determining other components of $u(x)$ to get a closed form solution **or** an approximate solution as discussed earlier.

Moreover, it is important to note that, even though this is a remarkable achievement that speeds the convergence of the solution and minimizes the size of calculations work, but unfortunately the self cancelling noise terms do not appear always, but a necessary condition should hold for the possibility that these terms may appear. The condition of the appearance of the noise terms requires the existence of the exact solution as one term of the zeroth component $u_0(x)$.

In the following we discuss some examples which illustrate the above outlined decomposition scheme, the modified decomposition method, and the phenomenon of the self cancelling noise terms as well.

Example 1: Solve the following Fredholm integro-differential equation

$$u'(x) = cos(x) + \frac{1}{4}x - \frac{1}{4}\int_0^{\frac{\pi}{2}} xtu(t)\,dt\,, \qquad u(0) = 0. \tag{5.166}$$

Solution: Integrating both sides of the equation (5.166) from 0 to x gives

$$u(x) - u(0) = sin(x) + \frac{1}{8}x^2 - \frac{1}{8}x^2 \int_0^{\frac{\pi}{2}} tu(t)\, dt\,, \qquad u(0) = 0. \qquad (5.167)$$

which gives upon using the initial condition

$$u(x) = sin(x) + \frac{1}{8}x^2 - \frac{1}{8}x^2 \int_0^{\frac{\pi}{2}} tu(t)\, dt\,. \qquad (5.168)$$

(i) *Using the Adomian decomposition method:* We usually decompose the solution into a series form given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x)\,. \qquad (5.169)$$

Substituting Eq. (5.169) into both sides of Eq. (5.168) yields

$$\sum_{n=0}^{\infty} u_n(x) = sin(x) + \frac{1}{8}x^2 - \frac{1}{8}x^2 \int_0^{\frac{\pi}{2}} t\left(\sum_{n=0}^{\infty} u_n(t)\right) dt\,. \qquad (5.170)$$

The Adomian decomposition method admits the use of the recurrence relation

$$\left.\begin{aligned} u_0(x) &= sin(x) + \frac{1}{8}x^2\,, \\ u_1(x) &= -\frac{1}{8}x^2\left(\int_0^{\frac{\pi}{2}} h(t)\,u_0(t)dt\right) = -\frac{1}{8}x^2 - \frac{\pi^4}{16^3}x^2\,, \\ u_2(x) &= -\frac{1}{8}x^2\left(\int_0^{\frac{\pi}{2}} h(t)\,u_1(t)dt\right) = \frac{\pi^4}{16^3}x^2 + \frac{\pi^8}{2\times 16^5}x^2\,, \\ u_3(x) &= -\frac{1}{8}x^2\left(\int_0^{\frac{\pi}{2}} h(t)\,u_2(t)dt\right) = -\frac{\pi^8}{2\times 16^5}x^2 + other\ terms\,, \\ &= \cdots \end{aligned}\right\} \quad (5.171)$$

In view of these results, we obtain the exact solution

$$u(x) = sin(x)\,. \blacksquare \quad (5.172)$$

(ii) *Using the noise terms phenomenon*: Using the first two components $u_0(x)$ and $u_1(x)$,we observe that the two identical terms $\frac{1}{8}x^2$ appear in these two components with opposite signs. Cancelling this noise term from $u_0(x)$, the remaining non-cancelled term in $u_0(x)$ gives the exact solution

$$u(x) = sin(x)\,. \blacksquare \quad (5.173)$$

that should be checked if it justifies the integro-differential equation.

(iii) *Using the modified decomposition method*: We first decompose the data function

$f(x) = sin(x) + \frac{1}{8}x^2$ as follows:

$$\left.\begin{aligned} f_0(x) &= sin(x)\,, \\ f_1(x) &= \frac{1}{8}x^2\,, \end{aligned}\right\} \quad (5.174)$$

using the modified recurrence relation (5.165) gives

$$\left.\begin{aligned} u_0(x) &= sin(x)\,, \\ u_1(x) = \frac{1}{8}x^2 - \frac{1}{8}x^2\left(\int_0^{\frac{\pi}{2}} t\,u_0(t)dt\right) &= 0\,, \end{aligned}\right\} \tag{5.175}$$

that gives the exact solution

$$u(x) = sin(x)\,. \blacksquare \tag{5.176}$$

Example 2: Solve the following Fredholm integro-differential equation

$$u'''(x) = sin(x) - x - \int_0^{\frac{\pi}{2}} xtu'(t)\,dt\,, \qquad u(0) = 1, u'(0) = 0, u''(0) = -1, \tag{5.177}$$

Solution: *Using the Adomian decomposition method:* We point out here that the first derivative $u'(x)$ of the unknown function $u(x)$ appears under the integral sign in this example. The approach we will follow is the same as used before, and will be illustrated through the solution. Integrating both sides of the equation (5.177) three times from 0 to x yields

$$u(x) - u(0) - \frac{1}{2!}x^2u''(0) = cos(x) + \frac{1}{2!}x^2 - \frac{1}{4!}x^4 - 1 - \frac{1}{4!}x^4\int_0^{\frac{\pi}{2}} tu'(t)\,dt\,, \tag{5.178}$$

which gives upon using the initial conditions in Eq. (5.177) .

$$u(x) = cos(x) - \frac{1}{4!}x^4 - \frac{1}{4!}x^4\int_0^{\frac{\pi}{2}} tu'(t)\,dt\,. \tag{5.179}$$

We begin by expressing the solution *u*(*x*) into a series form by

$$u(x) = \sum_{n=0}^{\infty} u_n(x)\,. \tag{5.180}$$

Substituting Eq. (5.180) into both sides of Eq. (5.179) yields

$$\sum_{n=0}^{\infty} u_n(x) = cos(x) - \frac{1}{4!}x^4 - \frac{1}{4!}x^4 \int_0^{\frac{\pi}{2}} t\left(\sum_{n=0}^{\infty} u_n(t)\right) dt\,. \qquad (5.181)$$

Proceeding as before, we determine the components $u_i(x), i \geq 0$ recurrently, where we set

$$u_0(x) = cos(x) - \frac{1}{4!}x^4\,, \qquad (5.182)$$

$$u_1(x) = -\frac{1}{4!}x^4\left(\int_0^{\frac{\pi}{2}} h(t)\,u_0'(t)dt\right) = \frac{1}{4!}x^4 + \frac{\pi^5}{(5!)(3!)(32)}x^4\,, \qquad (5.183)$$

$$u_2(x) = -\frac{1}{4!}x^4\left(\int_0^{\frac{\pi}{2}} h(t)\,u_1'(t)dt\right) = -\frac{\pi^5}{(5!)(3!)(32)}x^4 \qquad (5.184)$$

$$\vdots$$

Considering the first two components $u_0(x)$ and $u_1(x)$, we observe that the two identical terms $\frac{1}{4!}x^4$ appear in these components with opposite signs. Cancelling these terms and justifying that the remaining non-cancelled term of $u_0(x)$ satisfies the given equation yield the exact solution

$$u(x) = cos(x)\,. \blacksquare \qquad (5.185)$$

Notice that we can solve this example by applying the *modified decomposition method*. We leave it as an exercise to use this technique.

► 5.3.3 The Variational Iteration Method (VIM)

The variational iteration method **[7–11]** was presented in **Chapters 3 and 4** in detail, therefore we introduce a summary of the main steps. The method usually computes the successive approximations that will converge to the exact solution if such a solution exists.

Otherwise, the obtained approximations can be sued for numerical computations especially if the problem is concrete and an exact solution cannot be obtained.

The standard *j*th order Fredholm integro-differential equation reads

$$u^{(j)}(x) = f(x) + \int_a^b K(x,t)u(t)\,dt\,, \qquad (5.186)$$

Where $u^{(j)}(x) = \frac{d^j u}{dx^j}$ and $u(0), u'(0), u''(0), \dots, u^{j-1}(0)$ are the initial conditions that should be given.

The correction functional for the integro-differential equation Eq. (5.186) is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(x,t)\left(u_n^{(i)}(t) - f(t) - \int_a^b K(t,r)u_n(r)\,dr \right) dt\,. \qquad (5.187)$$

To apply the VIM, we should first determine the Lagrange multiplier $\lambda(t)$ that can be identified optimally as presented in **Chapters 3 and 4**.

Having $\lambda(t)$ determined, an iteration formula should be used for the determination of the determination of the successive approximations $u_{n+1}(x)\,, n \geq 0$ of the solution $u(x)$. The zeroth approximation $u_0(x)$ can be any selective function, but using the given initial values $u(0), u'(0), u''(0), \dots$ are usually used as examined earlier. Consequently the solution is given by

$$u(x) = \lim_{n\to\infty} u_n(x)\,. \qquad (5.188)$$

It is interesting to recall that the VIM evaluates successive approximations of the solution and not the components of the solution as in ADM. The last obtained approximation can be considered as $u_n(x)$ where its limit should be evaluated as given earlier.

In what follows, the VIM will be examined by investigating the following examples.

The VIM will be illustrated by studying the following example.

Example 1: Solve the Fredholm integro-differential equation by using the VIM.

$$u'(x) = 2 - sin(x) + \int_0^\pi tu(t)\,dt, \qquad u(0) = 1\,. \qquad (5.189)$$

Solution: The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - 2 + sin(t) - \int_0^\pi r u_n(r)\, dr \right) dt\,, \quad (5.190)$$

where we used $\lambda = -1$ for first-order integro-differential equations. Notice that the correction functional involves Volterra and Fredholm integral equations.

We can use the initial condition to select $u_0(x) = u(0) = 1$. Using this selection into the correction functional gives the following successive approximations

$$u_0(x) = 1\,, \quad (5.191)$$

$$u_1(x) = u_0(x) - \int_0^x \left(u_0'(t) - 2 + sin(t) - \int_0^\pi r u_0(r)\, dr \right) dt$$

$$= cos(x) + 2x + \frac{\pi^2}{2} x\,, \quad (5.192)$$

$$u_2(x) = u_1(x) - \int_0^x \left(u_1'(t) - 2 + sin(t) - \int_0^\pi r u_1(r)\, dr \right) dt$$

$$= (cos(x) + 2) + \left(-2x + \frac{\pi^2}{2} x \right) + \left(-\frac{\pi^2}{2} x + \frac{2\pi}{3} x \right) + other\ terms\,, \quad (5.193)$$

$$u_3(x) = u_2(x) - \int_0^x \left(u_2'(t) - 2 + sin(t) - \int_0^\pi r u_2(r)\, dr \right) dt$$

$$= (cos(x) + 2) + \left(-2x + \frac{\pi^2}{2} x \right) + \left(-\frac{\pi^2}{2} x + \frac{2\pi}{3} x \right)$$

$$+\left(-\frac{2\pi}{3}x + other\ terms\right), \tag{5.194}$$

and so on. The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x)\ . \tag{5.195}$$

It is obvious that noise terms appear in the successive approximations, that will be cancelled in the limit. Hence, we obtain the exact solution

$$u(x) = cos(x)\ .\ \blacksquare \tag{5.196}$$

► 5.3.4 Converting to Fredholm Integral Equations

In this section we will discuss a technique that will convert Fredholm integro-differential equation to an equivalent Fredholm integral equation. This can be easily done by integrating both sides of the integro-differential equation as many times as the order of the derivative involved in the equation from 0 to x for every time we integrate, and by using the given initial conditions.

It is important to note that this technique is applicable only if the Fredholm integro-differential equation involves the unknown function $u(x)$ only, and not any of its derivatives, under the integral sign.

Having established the transformation to a standard Fredholm integral equation, we can use any of the methods that were discussed before in **Chapter 4**, namely the decomposition method, the variational iteration method, the direct computation method, the successive approximations method or the successive substitutions method.

To give a clear overview of this method we discuss the following illustrative examples.

Example 1: Solve the following Fredholm integro-differential equation by converting it to a standard Fredholm integral equation.

$$u'(x) = 1 - \frac{1}{3}x\ + x\int_0^1 tu(t)\,dt, \qquad u(0) = 0\ . \tag{5.197}$$

Solution: Integrating both sides from 0 to x and using the initial condition we obtain

$$u(x) = x - \frac{1}{3!}x^2 + \frac{1}{2!}x^2\left(\int_0^1 tu(t)\,dt\right). \qquad (5.198)$$

It can be easily seen that Eq. (5.198) is a Fredholm integral equation; therefore we can select any method that was introduced earlier. To achieve our goal, we select the *successive approximations method* to solve this equation. Hence we set a zeroth approximation by

$$u_0(x) = x\,, \qquad (5.199)$$

and using this choice in Eq. (5.198) yields the first approximation

$$u_1(x) = x - \frac{1}{3!}x^2 + \frac{1}{2!}x^2\left(\int_0^1 t^2\,dt\right) = x\,. \qquad (5.200)$$

It is obvious that if we continue in the same manner, we then obtain

$$u_n(x) = x\,. \qquad (5.201)$$

Accordingly,

$$u(x) = \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} x = x\,. \quad \blacksquare \qquad (5.202)$$

Example 2: Solve the following Fredholm integro-differential equation by converting it to a standard Fredholm integral equation.

$$u''(x) = e^x - x + x\int_0^1 tu(t)\,dt, \qquad u(0) = 1\,, \qquad u'(0) = 1. \qquad (5.203)$$

Solution: Integrating both sides of Eq. (5.203) twice from 0 to x and using the initial conditions we obtain

$$u(x) = e^x - \frac{1}{3!}x^3 + \frac{1}{3!}x^3\left(\int_0^1 tu(t)\,dt\right). \qquad (5.204)$$

It can be easily seen that Eq. (5.204) is a Fredholm integral equation; therefore we can select any method that was introduced earlier. To achieve our goal, we select *the direct computation method* for this Example. Therefore we can express Eq. (5.204) in the form

$$u(x) = e^x - \frac{1}{3!}x^3 + \alpha\frac{1}{3!}x^3, \tag{5.205}$$

where the constant α is defined by the definite integral

$$\alpha = \int_0^1 tu(t)\,dt\,. \tag{5.206}$$

Substituting Eq. (5.205) into Eq. (5.206) we obtain

$$\alpha = \int_0^1 t\left(e^t - \frac{1}{3!}t^3 + \alpha\frac{1}{3!}t^3\right)dt\,. \tag{5.207}$$

an easy integral to evaluate, from which we obtain

$$\alpha = 1\,. \tag{5.208}$$

Inserting the value of α obtained in Eq. (5.208) into Eq. (5.205) yields the exact solution given by

$$u(x) = e^x\,. \quad \blacksquare \tag{5.209}$$

Remark 5.2: It is worth noting that the main ideas we applied are the direct computation method, the decomposition method, and the VIM, where the noise term phenomena was introduced.

The direct computation method provides the solution in a closed-form, but the decomposition method provides the solution in a rapidly convergent series, where the VIM evaluates successive approximations of the solution and **not** the components of the solution as in ADM. The last obtained approximation can be considered as $u_n(x)$ where its limit should be evaluated.

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